Journal of Mathematical Physics

Construction of a complex-valued fractional Brownian motion of order *N*

Cite as: J. Math. Phys. **33**, 3128 (1992); https://doi.org/10.1063/1.529976
Submitted: 03 August 1989 • Accepted: 17 February 1992 • Published Online: 04 June 1998

Philippe Sainty

Construction of a complex-valued fractional Brownian motion of order *N*

Philippe Sainty
Université Pierre et Marie Curie, Mathématiques, tour 45-46, 5 ème étage, 4, Place Jussieu, 75252 Paris Cedex 05, France

(Received 3 August 1989; accepted for publication 17 February 1992)

In this paper, a Brownian motion of order n $(n>2)$ is defined by a probabilistic approach different from Hochberg's and Mandelbrot's. This process is constructed from sums of independent $\mathbb{R}_+^{1/n}$-valued random variables (rv) (where $\mathbb{R}_+^{1/n}=\{z\in\mathbb{C};\ z^n\in\mathbb{R}_+\}$). Many properties of the real standard Brownian motion are generalized at order n, but in the case $n>2$, it is interesting to describe the Brownian motion of order n on the σ algebra $\otimes[B(\mathbb{R}_+^{1/n})]^{\mathbb{R}_+}$ [where $B(\mathbb{R}_+^{1/n})$ is the σ algebra generated by sets of type $A(0,h)=\{z\in\mathbb{C};\ z^n\in[0;h^n[\},(h\in\mathbb{R}_+^*)$]. This σ algebra is totally different from $\otimes[B(\mathbb{R})]^{\mathbb{R}_+}$. Thus this study shows the fractal nature of the Brownian motion of order n, and given invariance scale (self-similarity) properties. Then, a stochastic integral and an Itô–Taylor lemma at order n are given to allow the representation of the solution of the heat equation of order n by a probabilistic average. All these results can be obtained via nonstandard analysis methods (infinitesimal time discretization). Finally, one remarks that this process has a.s (almost surely) continuous sample paths, infinite variance, and independent increments, whereas the fractional Brownian motion of Mandelbrot has a.s continuous sample paths, finite variance, and interdependent increments.

I. INTRODUCTION

All the shown results come from the properties of the complex roots of the unity of order n and the set $\mathbb{R}_+^{1/n}=\{z\in\mathbb{C};z^n\in\mathbb{R}_+\}$. We want to study the properties of the heat equation of order n (1) by a probabilistic way generalizing Itô's calculus,

$$\frac{\partial u}{\partial t}(x,t)=\frac{1}{n!}\cdot\frac{\partial^n u}{\partial x^n}(x,t),\quad (x,t)\in\mathbb{R}x\mathbb{R}_+^*;$$
$$u(x,0)=f(x),\quad (x\in\mathbb{R});\tag{1}$$

with f a bounded and continuous given $(\mathbb{R}\to\mathbb{R})$ function.

Following Block,[1] this problem is essentially linked up to the following ODE of order $n-1$ by using the Fourier transform on (1). Then, the fundamental solution of (1) verifies (2):

$$\frac{d^{n-1}w}{dz^{n-1}}(z)+(n-1)!\cdot z\cdot w(z)=0,\tag{2}$$

where $w(0),w^{(1)}(0),...,w^{(n-2)}(0)$ are given real constants.

With the help of the notion of generalized pictures developed by Block,[1] one remarks that (1) is invariant on $\mathbb{R}_+^{1/n}$. That is right, the operators d/dt and $(1/n!)\cdot d^n/dx^n$ accept, respectively, $\exp(\lambda t)$ and $\exp[(n!\lambda)^{1/n}\cdot\omega_k\cdot x]$, $(\omega_k=\exp(2i\pi k/n);\ k=0,...,n-1)$ $(\lambda\in\mathbb{C})$ as a system of characteristic vectors. Then, to follow the basic construction of the Brownian motion, it seems necessary to study sums of independent $\mathbb{R}_+^{1/n}$-valued rv with the same law in each branch of $\mathbb{R}_+^{1/n}$; the result is $\mathbb{C}$ valued, and to have finite characteristics on this limit of rv sums, one will often examine the rv sums on the tensorial product σ algebra constructed from the σ algebra $B(\mathbb{R}_+^{1/n})$ generated by sets of type: $A(0,h)=\{z\in\mathbb{C};z^n\in[0,h^n[\},\ (h\in\mathbb{R}_+^*)$. Then, it reveals the fractal nature in the complex plane of the sample paths of this Brownian motion of order n $(n>2)$. Furthermore, one obtains a weak central limit theorem of order n like Hochberg's,[2] but in a true probabilistic context on the complex plane. Of course, the law of the rv sums is not infinitely divisible in the Levy–Doob sense.[3,4] Noting down $\mathscr{S}$ the obtained limit, one has the "strange" properties:

$$(\forall\lambda\in\mathbb{C}),$$
$$E\{\exp(\lambda\cdot\mathscr{S})\}=\exp(\lambda^n/n!)$$

and

$$E\{|\mathscr{S}|^2\}=+\infty.\tag{3}$$

This limit is linked up (via a definite isometry), to the cone

$$l_n(\mathbb{R}_+)=\left\{(\alpha_k)_{k\in\mathbb{N}};\quad \alpha_k\in\mathbb{R}_+;\quad \sum_{k=0}^{+\infty}\alpha_k^n<+\infty\right\}.$$

From that, with a linear cut approximation, one deduces the construction of a $\mathbb{C}$-valued process $X_{[n]}$, with stationary, independent, self-similar increments and a.s continuous sample paths definite from the σ algebra $\otimes[B(\mathbb{R}_+^{1/n})]^{\mathbb{R}_+}$, which is an infinite product of copies of

$B(\mathbb{R}_+^{1/n})$ (in the compactified Alexandrov's sense). $X_{[n]}$ verifies, as well, the following "strange" properties:

$$(\forall \quad 0\leqslant s<t<+\infty),$$

$$(\forall \lambda\in\mathbb{C}), \quad E\{|X_{[n]}(t)-X_{[n]}(s)|^2\}=+\infty$$

and

$$E\{\exp(\lambda\cdot[X_{[n]}(t)-X_{[n]}(s)])\}=\exp\left\{\frac{\lambda^n}{n!}\cdot(t-s)\right\}. \tag{4}$$

Finally, $X_{[n]}$ has several properties generalizing these known at order 2 (Brownian scaling, isotropy on the branches of $\mathbb{R}_+^{1/n}$, no differentiability of the sample continuous paths, inversion at order n, Levy's continuity Modulus of paths with bounded nth-order variations, etc.). From that, one defines stochastic integrals of the following type:

$$\int_0^t f[X_{[n]}(s)]\cdot(dX_{[n]}(s))^k \quad (1\leqslant k\leqslant n), \tag{5}$$

for $f\in L_n(d\mathbf{P}\times dt)$ or f having an analytic prolongation to $\mathbb{C}$, with "good" properties at infinity. Here (5) is defined from a prolongation of an isometry on the cone $l_n(\mathbb{R}_+)$ via the strictly definite positive functional $E^{1/n}\{(\cdot)^n\}$. Then, via the Taylor formula developed at order n, one obtains an Itô–Taylor lemma at order n for a certain class of analytic functions on $\mathbb{C}$,

$$\begin{aligned} d[f(X_{[n]}(t))] &= \sum_{k=1}^{n} \frac{1}{k!}\cdot f^{(k)}(X_{[n]}(t))\cdot[dX_{[n]}(t)]^k \\ &= \left\{\exp\left(dX_{[n]}(t)\cdot\frac{d}{dz}\right)-Id\right\}\cdot(f(X_{[n]}(t))) \\ &(t\geqslant 0) \end{aligned} \tag{6}$$

(it is a random drift in the operational sense).

It is then easy to deal with the solution of the heat equation of order n and his representation via the notion of heat or Hermite polynomials of order n,[5,6] and general problems (Feynman–Kac problems at order n)[7,8] studied directly in Ref. 9 with the help of Trotter's formula. Finally, a study of basic measures implicitly used in this paper to define the Brownian motion of order n is given, and one links together the Hochberg's "probability"[2] and that described in this paper. This relation is obtained from an extension of the Fourier transform, because the support of $X_{[n]}$ is of fractal nature, whereas the support of Hochberg's is real. Many properties of $X_{[n]}$ are totally different from those of the Lévy–Mandelbrot process.[10] One can equally remark that all these results can be showed via nonstandard analysis methods following Keisler[11] with his nonstandard Itô's calculus or Berger and Sloan[12] with their nonstandard Hochberg's calculus.

II. *N*-GAUSSIAN RANDOM VARIABLES

Let $(\Omega,F,\mathbb{P})$ denote a reference probability space and n a natural number ($n>2$; the cases $n=1,2$ are known). Noting down $\omega_k=\exp(2i\pi k/n)$ for $k=0,\ldots,n-1$; then one has the following results.

Definition 2.1: One calls Rademacher rv of order n, all discrete rv $\epsilon_{[n]}$, with a uniform distribution in $\Omega=\{\omega_k; k=0,\ldots,n-1\}$, i.e.,

$$\mathbb{P}\{\epsilon_{[n]}=\omega_k\}=1/n, \quad \text{for } k=0,1,\ldots,n-1. \tag{7}$$

Lemma 2.2: Denote by $\delta_{q,0}$ the Kronecker's symbol. Then one has

$$(\forall \quad p\in\mathbb{N}), \ (\forall \quad q=0,\ldots,n-1), \ E\{\epsilon_{[n]}^{np+q}\}=\delta_{q,0}, \tag{8}$$

$$\begin{aligned}(\forall \quad \lambda\in\mathbb{C}), \quad E\{\exp(\lambda\cdot\epsilon_{[n]})\}&=\cosh_{[n]}(\lambda) \\ &=\sum_{k=0}^{+\infty}\frac{\lambda^{nk}}{(nk)!},\end{aligned} \tag{9}$$

where $\cosh_{[n]}$ is the hyperbolic cosine function of order n.[13–15]

Proof: It results directly from the properties of the complex roots of the unity of order n, because of the following calculus:

$$\begin{aligned} E\{\epsilon_{[n]}^{np+q}\}&=n^{-1}\cdot\left(\sum_{k=0}^{n-1}\omega_k^{np+q}\right) \\ &=n^{-1}\cdot\left(\sum_{k=0}^{n-1}\omega_{np+q}^{k}\right)=\delta_{q,0} \end{aligned}$$

(that is a geometric series), therefore the result.

Notation 2.3: One notes that

$$E_{1/n}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(1+k/n)} \quad (z\in\mathbb{C}) \tag{10}$$

[with $\Gamma(x)=\int_0^{+\infty}e^{-t}\cdot t^{x-1}\cdot dt \quad (x>0)$], the Mittag–Leffler function of order n.[16–18]

Properties 2.4: From Refs. 16, 17, and 19, one knows that $E_{1/n}(-x)\quad(x\geqslant 0)$ is a function totally monotonic.

Definition 2.5: Note down X_n, the real positive rv defined by his Laplace transform, i.e.,

$$(\forall \quad \lambda\geqslant 0), \quad E\{\exp(-\lambda\cdot X_n)\}=E_{1/n}(-\lambda/K_n),$$

$$(K_n=(n!)^{1/n}), \tag{11}$$

and $\epsilon_{[n]}$ a Rademacher rv of order n independent of X_n; then, one calls standard n-Gaussian rv, all $\mathbb{R}_+^{1/n}$-valued rv noting that $G_{[n]}$ with a law verifying

$$G_{[n]} \overset{\text{Law}}{\sim} \epsilon_{[n]} \cdot X_n \quad (\text{one remarks that: } |G_{[n]}| \overset{\text{Law}}{\sim} X_n). \tag{12}$$

Remark 2.6: The density of the $|G_{[n]}|$'s law has been given before by Pollard[19] and Zolotarev.[18] It has the following expression:

$$p_n(x) = \frac{n}{\pi} K_n \cdot \left\{ \sum_{k=1}^{+\infty} \frac{\sin(k\pi/n)}{k!} \cdot \Gamma\left(1+\frac{k}{n}\right) \cdot [-K_n x]^{k-1} \right\}. \tag{13}$$

Lemma 2.7: One has the following relation:

$$|G_{[n]}| \overset{\text{law}}{\sim} [n(!)Z_n]^{-1/n} \tag{14}$$

(where Z_n is a $1/n$-stable real positive rv normalized by his Laplace transform), i.e.,

$$(\forall \quad \mu \geqslant 0), \quad E\{\exp(-\mu \cdot Z_n)\} = \exp(-\mu^{1/n}). \tag{15}$$

Proof: See Refs. 20 and 18 for the proof (the duality relation between Z_n and $|G_{[n]}|$).

Remark 2.8: For example, at order 4, one gets

$$|G_{[4]}| \overset{\text{Law}}{\sim} (3)^{-1/4} \cdot \sqrt{|G|} \cdot |G'|, \tag{16}$$

where G and G' are two independent real standard Gaussian rv; this remark is used implicitly in Ref. 9 for the treatment of the Feynman–Kac formula at order 4.

Lemma 2.9: Here p_n, (see Ref. 13) has, equally, the following expression:

$$p_n(x) = \frac{n^2}{\pi x} K_n \int_0^{+\infty} t^{n-1} \exp\left\{ -t^n - K_n \cos\left(\frac{2\pi}{n}\right) xt \right\} \sin\left\{ K_n \sin\left(\frac{2\pi}{n}\right) xt \right\} dt. \tag{17}$$

Proof: This expression is a Hardy function,[18] and one can see that there exists an inversion relation (called a duality relation by Zolotarev[18]), between the $1/n$-stable rv and $|G_{[n]}|$. To obtain the result, it is sufficient to write

$$\Gamma\left(1+\frac{k}{n}\right) = \int_0^{\infty} e^{-t} \cdot t^{k/n} \cdot dt = n \int_0^{+\infty} e^{-u^n} \cdot u^{k+n-1} \cdot du,$$

and then to permute the symbols $\int$ and Σ without any problems.

Remark 2.10: This formulation is near Hochberg's density,[2] which is written for n even as

$$P_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left\{ -ix\xi - \frac{\xi^n}{n!} \right\} d\xi = \frac{1}{\pi} \int_0^{+\infty} \exp\left(\frac{-z^n}{n!} \right) \cos(xz) dz. \tag{18}$$

Lemma 2.11: Let $G_{[n]}$ be a standard n-Gaussian rv; then

$$(\forall \quad \lambda \in \mathbb{C}), \quad E\{\exp(\lambda G_{[n]})\} = \exp(\lambda^n/n!). \tag{19}$$

Let $(G_{[n]}^{(k)})_{k \in \mathbb{N}^*}$ be a family of independent standard n-Gaussian rv and $(\alpha_k)_{k \in \mathbb{N}^*}$ a sequence of the cone

$$l_n(\mathbb{R}_+) = \left\{ (\alpha_k)_{k \in \mathbb{N}^*}; \alpha_k > 0; \sum_{k=1}^{+\infty} \alpha_k^n < +\infty \right\}.$$

Let $S_p = \Sigma_{k=1}^p \alpha_k G_{[n]}^{(k)}$; then S_p has the following remarkable properties:

$$S_p \overset{\text{Law}}{\sim} \omega_k S_p, \quad \text{for } k=0,\ldots,n-1. \tag{20a}$$

$$(\forall \quad \lambda \in \mathbb{C}), \ E\{\exp(\lambda S_p)\} = \exp\left\{ \frac{\lambda^n}{n!} \left(\sum_{k=1}^{p} \alpha_k^n \right) \right\} = \exp\left\{ \frac{\lambda^n}{n!} E(S_p^n) \right\}. \tag{20b}$$

But, if

$$\lim_{p \to +\infty} \sum_{k=1}^{p} \alpha_k^2 = +\infty,$$

then

$$\lim_{p \to +\infty} E\{|S_p|^2\} = +\infty \tag{20c}$$

($\mathbb{C}$ is too big a support to correctly analyze the S_p's properties; the S_p's support is of a fractal nature).

Proof: From (12), one has $G_{[n]} \overset{\text{Law}}{\sim} \epsilon_{[n]} X_n$; by using 2.5 and (10), one gets

$$(\forall \quad p \in \mathbb{N}), \ E\{X_n^{np}\} = (np)!/[p!(n!)^p].$$

Then

$$E\{G_{[n]}^{np+q}\}=E\{\epsilon_{[n]}^{np+q}\}E\{X_n^{np+q}\}$$

$$=\delta_{q,0}E\{X_n^{np}\}$$

$$=\delta_{q,0}\frac{(np)!}{p!(n!)^p},\quad \text{for all } q=0,...,n-1.$$

From that, one easily gets (19) by writing the analytic expression of exp. By construction (20a) is trivial. Equation (20b) results from (19), from the independence of the $G_{[n]}^{(k)}$ and from the fact that $E\{G_{[n]}^q\}=0$ for $q=0,...,n-1$. For Eq. (20c) one has $E\{|S_p|^2\}=\Sigma_{k=1}^{p}\alpha_k^2$; therefore the result.

Lemma 2.12: In the case when n is even, i.e., $n=2p$, ($p\in\mathbf{N}^*$), one links together P_{2p} and p_{2p} by the following "weak" identity.

For all function φ in the real Schwartz class (φ has an analytic extension to $\mathbb{C}$),

$$E\{\varphi(G_{[n]})\}=E\{\varphi(G_{[n]}^{(H)})\}$$

$$=\frac{1}{2\pi}\int_{\mathbf{R}}\hat{\varphi}(\lambda)\exp\left(\frac{-\lambda^n}{n!}\right)d\lambda, \tag{21}$$

where $\hat{\varphi}(\lambda)=\int_{\mathbf{R}}\exp(i\lambda x)\varphi(x)dx$ and $G_{[n]}^{(H)}$ is the Hochberg "rv," corresponding to $P_{2p}{}^2$].

Proof: Since $E\{\exp(\lambda G_{[n]})\}=\exp(\lambda^n/n!)$, one gets directly the result by extension of the Parseval identity on the branches of $\mathbf{R}_+^{1/n}$ (the idea of the proof is the same as for Lemma 2.15). One remarks that the σ algebra associated with $G_{[n]}$ is generated by the sets $A(0,h)$ and that $G_{[n]}$ has a probabilistic sense, whereas $G_{[n]}^{(H)}$ is defined on $B(\mathbf{R})$, and has a signed density. Thus, one has a weak identity compensating the difference of the laws by different supports.

Lemma 2.13: Let f denote a bounded and continuous ($\mathbf{R}\to\mathbf{R}$) function, allowing an analytic extension to $\mathbb{C}$, $\widetilde{f}$, which verifies

$$(\exists\quad A,M>0),\quad |\widetilde{f}(z)|\leqslant M\exp(A|z|^{n'})$$

$$(n^{-1}+n'^{-1}=1), \tag{22}$$

i.e., $\widetilde{f}$ is of order n'; then

$$\lim_{h\to 0}h^{-1}\cdot[E\{\widetilde{f}(z+z_0h^{1/n}G_{[n]})\}-\widetilde{f}(z)]=\frac{z_0^n}{n!}\frac{d^n}{dz^n}\widetilde{f}(z). \tag{23}$$

More generally, for a holomorphic semigroup $\exp(zA)$, one gets

$$\{\exp(z_0A^n)\}[\widetilde{f}(z)]=\langle\exp(z_0^{1/n}G_{[n]}A),\widetilde{f}(z)\rangle, \tag{24}$$

and for an analytic function φ in $\mathbb{C}$,

$$\{\exp(z_0\varphi(A))\}[\widetilde{f}(z)]$$

$$=\left\langle\exp\left[\left(\sum_{k=1}^{+\infty}\left(z_0\frac{\varphi^{(k)}}{k!}(0)\right)^{1/k}G_{[k]}^{(k)}\right)A\right],\right.$$

$$\left.\widetilde{f}(z+z_0\varphi(0))\right\rangle \tag{25}$$

(with a formal sense), where $(G_{[k]}^{(k)})_{(k\in\mathbf{N}^*)}$ is a sequence of independent standard k-Gaussian rv (in particular, $G_{[1]}^{(1)}$ and $G_{[2]}^{(2)}$ is a standard Gaussian rv), and A is the infinitesimal generator of $\exp(zA)$.

Proof ($\forall h,z,z_0\in\mathbb{C}$):

$$h^{-1}(E\{\widetilde{f}(z+z_0h^{1/n}G_{[n]})\}-\widetilde{f}(z))$$

$$=h^{-1}\left(z_0h^{1/n}E\{G_{[n]}\}\widetilde{f}'(z)+\cdots\right.$$

$$\left.+z_0^n\frac{h}{n!}E\{G_{[n]}^n\}\widetilde{f}^{(n)}(z)+z_0^{n+1}\frac{h^{1+1/n}}{(n+1)!}O(1)\right);$$

since $E\{G_{[n]}^{np+q}\}=\delta_{q,0}[(np)!/(p!(n!)^p)]$, for all $p\in\mathbf{N}$ and $q=0,...,n-1$, one gets

$$\lim_{h\to 0}h^{-1}[E\{\widetilde{f}(z+z_0h^{1/n}G_{[n]})\}-\widetilde{f}(z)]=z_0^n\frac{\widetilde{f}^{(n)}}{n!}(z).$$

For the generalization to a semigroup, the same arguments are used. Then, one studies the semigroup $\exp\{(t/n!)d^n/dx^n\}$ from the examination of a central limit theorem at order n and a stochastic process generalizing the standard real Brownian motion.

Lemma 2.14: Let $(\alpha_k)_{k\in\mathbf{N}^*}$ denote a sequence of positive real numbers in the cone $l_n(\mathbf{R}_+)=\{(\alpha_k)_{k\in\mathbf{N}^*}:\Sigma_{k=1}^{+\infty}\alpha_k^n<+\infty\}$, $(G_{[n]}^{(k)})_{k\in\mathbf{N}^*}$ a family of independent standard n-Gaussian rv and S_p the sum defined by $S_p=\Sigma_{k=1}^{p}\alpha_kG_{[n]}^{(k)}$; then, $w\text{-}\lim_{p\to+\infty}S_p$ exists in the following sense: $\lim_{p\to+\infty}E\{\widetilde{\varphi}(S_p)\}$ exists for all functions φ, verifying the following conditions: φ is in $L_1(\mathbb{R},dx)$, continuous, with an analytic extension $\widetilde{\varphi}$ to $\mathbb{C}$, such that

$$\left(\forall\quad 0\leqslant x\leqslant\sum_{k=1}^{+\infty}\alpha_k^n\right),$$

$$\hat{\varphi}(\lambda)\exp\{(-i\lambda)^nx/n!\}\in L_1(\mathbb{R},d\lambda)\quad(\lambda\in\mathbb{R}).$$

Proof: The set of φ functions is denoted by $H_{n'}(\mathbb{C})$, and in all the cases one can choose $\widetilde{\varphi}$ of order $n'=n/(n-1)$. This kind of convergence is an extension of the usual weak convergence.[21,22] For the proof, one comes down to the Stone–Weierstrass theorem of approximation of an

analytic function on $\mathbb{C}$ by a sequence of polynomials, $(P_l)_{l\in\mathbb{N}}$ in the sense of the uniform convergence on a compact set of $\mathbb{C}$.

Let $(0<q<p)$, $(p,q\in\mathbb{N}^*)$, and $(a\in\mathbb{R}_+^*)$. Let $B_a=\{z\in\mathbb{C};|z|\leqslant a\}$, and let T_a be the stopping time defined by $T_a=\{\inf k>0,\ (k\in\mathbb{N}^*)\colon\ |S_k|\geqslant a\}$; then one gets $|S_{p\wedge T_a}|\leqslant a$, and since $\lim_{a\to+\infty}T_a=+\infty$, it results that $|E\{\tilde{\varphi}(S_p)\text{-}\tilde{\varphi}(S_q)\}|\leqslant M$ (because $\forall p\in\mathbb{N}^*$, $\lim_{a\to+\infty}S_{p\wedge T_a}=S_p$), and we also come down to a "compact problem" on $S_{p\wedge T_a}$. Furthermore, $\hat{\varphi}(\lambda)\exp\{(-i\lambda)^n x/n!\}\in L_1(\mathbb{R},d\lambda)$, and then, noting that $\hat{S}_p=S_p\wedge T_a$ ($\exists\ M_a$, $M>0$), such that $|E\{\tilde{\varphi}(\hat{S}_p)-\tilde{\varphi}(\hat{S}_q)\}|\leqslant M_a\leqslant M<+\infty$; also, one gets

$$(\forall\quad\epsilon>0)\quad(\exists l>0),$$

$$|E\{\tilde{\varphi}(\hat{S}_p)-\tilde{\varphi}(\hat{S}_q)\}|\leqslant 2\epsilon+|E\{P_l(\hat{S}_p)-P_l(\hat{S}_q)\}|$$

[because of the uniform convergence of P_l to $\tilde{\varphi}$, $(l\to+\infty)$, on the compact set of $\mathbb{C}$, B_a].

Furthermore, for $j=1,\dots,n-1$, $E\{(\hat{S}_p-\hat{S}_q)^j\}=0$ and

$$|E\{(\hat{S}_p-\hat{S}_q)^n\}|\leqslant\sum_{k=q}^{p-1}\alpha_k^n\to 0$$

$$(q\to+\infty)\quad((\alpha_k)_{k\in\mathbb{N}^*}\in l_n(\mathbb{R}_+)).$$

One works on a compact set (because of the stopping time T_a); for all $\varphi\in H_{n'}(\mathbb{C})$ ($\exists\quad M_\varphi>0$), $|E\{\tilde{\varphi}(S_p)|\leqslant M_\varphi<+\infty$; then

$$(\exists\quad K_a,K>0),\sup_{|z|\leqslant a}\left|\frac{d^n}{dz^n}P_l(z)\right|\leqslant K_a\leqslant K<+\infty.$$

Hence

$$(\forall\quad\epsilon,\epsilon'>0)\quad(\exists\quad q\in\mathbb{N}),$$

$$|E\{\tilde{\varphi}(\hat{S}_p)-\tilde{\varphi}(\hat{S}_q)\}|\leqslant 2\epsilon+K\epsilon',$$

by using the Taylor's development of P_l at order n from $\hat{S}_q$. One then deduces that $\lim_{p\to+\infty}E\{\tilde{\varphi}(\hat{S}_p)\}$ exists for all $a<+\infty$, and going to the limit $a\to+\infty$, one finally gets the result, i.e., $\lim_{p\to+\infty}E\{\tilde{\varphi}(S_p)\}$ exists.

Lemma 2.15: Under the same hypothesis as in Lemma 2.14, one gets

$$\lim_{p\to+\infty}E\{\tilde{\varphi}(S_p)\}=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\varphi}(\lambda)\exp\left\{\frac{(-i\lambda)^n}{n!}\left[\lim_{p\to+\infty}E\{S_p^n\}\right]\right\}d\lambda. \tag{26}$$

Proof: Let $\int_{\mathbb{R}_+^{1/n}}=(1/n\sum_{r_k=0}^{n-1}\int_{\mathbb{R}_+})$ and $p_n(x)$ be the density defined in (13). Then ($\forall\quad z\in\mathbb{C}$), $\tilde{\varphi}(z)=(1/2\pi)\int_{\mathbb{R}}\hat{\varphi}(\lambda)\exp(-i\lambda z)d\lambda$, and one deduces that

$$E\{\tilde{\varphi}(S_p)\}$$

$$=\left(\frac{1}{n}\sum_{r_1=0}^{n-1}\int_{\mathbb{R}_+}\right)\cdots\left(\frac{1}{n}\sum_{r_p=0}^{n-1}\int_{\mathbb{R}_+}\right)$$

$$\times\tilde{\varphi}\left(\sum_{k=1}^{p}\alpha_k\omega_{r_k}x_k\right)p_n(x_1)\cdots p_n(x_p)dx_1\cdots dx_p$$

$$=\int_{\mathbb{R}_+^{1/n}}\cdots\int_{\mathbb{R}_+^{1/n}}\left[\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\varphi}(\lambda)\right.$$

$$\left.\times\exp\left\{-i\lambda\left(\sum_{k=1}^{p}\alpha_k\omega_{r_k}x_k\right)\right\}d\lambda\right]$$

$$\times p_n(x_1)dx_1\cdots p_n(x_p)dx_p.$$

By permuting $\int_{\mathbb{R}_+^{1/n}}$ and $\int_{\mathbb{R}}$, one remarks that

$$\int_{\mathbb{R}_+^{1/n}}\exp\{-i\lambda\alpha_k\omega_{r_k}x_k\}p_n(x_k)dx_k$$

$$=E\{\exp(-i\lambda\alpha_k G_{[n]}^{(k)})\}$$

$$=\exp\{(-i\lambda\alpha_k)^n/n!\}.$$

Then

$$E\{\tilde{\varphi}(S_p)\}=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\varphi}(\lambda)\exp\left\{\frac{(-i\lambda)^n}{n!}\left(\sum_{k=1}^{p}\alpha_k^n\right)\right\}d\lambda.$$

Therefore the result, going to the limit $(p\to+\infty)$, since, $E\{S_p^n\}=\sum_{k=1}^{p}\alpha_k^n$. In fact, one has used a kind of convolution on the branches of $\mathbb{R}_+^{1/n}$, the Parseval identity, and the following result on S_p:

$$(\forall\quad\lambda\in\mathbb{C}),\ (\forall\quad p\in\mathbb{N}^*),$$

$$E\{\exp(\lambda S_p)\}=\exp\left\{\frac{\lambda^n}{n!}\left(\sum_{k=1}^{p}\alpha_k^n\right)\right\}.$$

Remark 2.16: It is easy to improve this result with a greater class of functions φ, verifying only (i) φ is a continuous function in $L_1(\mathbb{R},dx)$; (ii) ($\forall\quad 0\leqslant x\leqslant\sum_{k=1}^{+\infty}\alpha_k^n$, $\hat{\varphi}(\lambda)\exp\{(-i\lambda)^n x/n!\}\in L_1(\mathbb{R},d\lambda)$; and (iii) ($\exists\quad\delta>0$), such that φ allows an analytic extension $\tilde{\varphi}$ to $B(\delta)=\{z\in\mathbb{C};|z|<\delta\}$. These conditions give a sense to $E\{\tilde{\varphi}(S_{p\wedge T_a})\}$ $(0<a\leqslant\delta)$.

Lemma 2.17: In fact, we have

$$E\{\widetilde{\varphi}(S_p)\}=\frac{1}{2\pi}\int_{\mathbf{R}}\widehat{\varphi}(\lambda)\exp\left\{\frac{(-i\lambda)^n E\{S_p^n\}}{n!}\right\}d\lambda$$
$$=E\{\widetilde{\varphi}(E^{1/n}\{S_p^n\}G_{[n]})\}$$
$$=\lim_{q\to+\infty}\frac{1}{n}\left\{\sum_{k=0}^{n-1}E\{P_q(\omega_k E^{1/n}\{S_p^n\}|G_{[n]}|)\}\right\}. \tag{27}$$

The notations and conditions are those of Lemmas 2.14 and 2.15.

Proof: For the two first identities, the proof is given in Lemma 2.15. For the third, $(P_q)_{q\in\mathbb{N}}$ is a sequence of polynomials approximating uniformly $\widetilde{\varphi}$ on each compact set of $\mathbb{C}$ and $G_{[n]}\overset{\text{Law}}{\sim}\epsilon_{[n]}|G_{[n]}|$, therefore the result. An immediate application is the weak central limit at order n.

Lemma 2.18: $Z_p=p^{-1/n}(\Sigma_{k=1}^n G_{[n]}^{(k)})$ converges weakly $(p\to+\infty)$ for the functions of $H_{n'}(\mathbb{C})$. Thus, for all functions $\widetilde{\varphi}\in H_{n'}(\mathbb{C})$, one has

$$E\{\widetilde{\varphi}(Z_p)\}=\frac{1}{2\pi}\int_{\mathbf{R}}\widehat{\varphi}(\lambda)\exp\left\{\frac{(-i\lambda)^n}{n!}\right\}d\lambda$$
$$=\sum_{k=0}^{+\infty}\frac{\widetilde{\varphi}^{(nk)}}{k!(n!)^k}(0). \tag{28}$$

Proof:

$$(\forall\quad\lambda\in\mathbb{C}),\quad E\{\exp(\lambda Z_p)\}=\prod_{k=1}^{p}E\left\{\exp\left(\frac{\lambda G_{[n]}^{(k)}}{p^{1/n}}\right)\right\}$$
$$=\prod_{k=1}^{p}\exp\left\{\frac{\lambda^n}{(n!p)}\right\}$$
$$=\exp(\lambda^n/n!).$$

Then, by using the results of Lemma 2.17, one obtains the first identity. For the second identity, it is sufficient to develop the analytic expression of $\widetilde{\varphi}$ and to conclude by remarking that $E\{Z_p^{nq+r}\}=\delta_{r,0}[(nq)!/q!(n!)^q]$, for all $q\in\mathbb{N}$ and $r=0,...,n-1$. One remarks that $E\{\widetilde{\varphi}(Z_p)\}$ is independent of p, i.e., it is a stationary sequence if it verifies that $U_p=\Sigma_{k=0}^p\varphi_{(0)}^{(nk)}/[k!(n!)^k]$ is convergent $(p\to+\infty)$.

If $\widetilde{\varphi}\in H_{n'}(\mathbb{C})$, U_p is convergent. If $\widetilde{\varphi}\notin H_{n'}(\mathbb{C})$, one can give an extension to this result by using a singular integral to give a formal sense to

$$\frac{1}{2\pi}\int_{\mathbf{R}}\widehat{\varphi}(\lambda)\exp\left\{\frac{(-i\lambda)^n}{n!}\right\}d\lambda.$$

Corollary 2.19: $(\forall\quad\widetilde{\varphi}H_{n'}(\mathbb{C}))$; one obtains

$$E\{\widetilde{\varphi}(Z_p)\}=E\{\widetilde{\varphi}(G_{[n]})\}=E\{\widetilde{\varphi}_0(|G_{[n]}|)\}, \tag{29}$$

with $\widetilde{\varphi}_0(x)=(1/n)(\Sigma_{k=0}^{n-1}\widetilde{\varphi}(x\omega_k))$. Noting that $Z=w-\lim_{p\to+\infty}Z_p$, one obtains

$$(\forall\quad\lambda\in\mathbb{C}),\quad E\{\exp(\lambda Z)\}=\exp(\lambda^n/n!), \tag{30a}$$

$$E\{|Z|^2\}=+\infty, \tag{30b}$$

$$Z_p\overset{\text{Law}}{\sim}Y_p+G_{[n]}, \tag{30c}$$

where Y_p and $G_{[n]}$ are independent rv, Y_p being $\mathbb{C}$ valued and verifying

$$w-\lim_{p\to+\infty}Y_p/p^{(n-2)/(2n)}=\frac{1}{\sqrt{2}}E^{1/2}\{|G_{[n]}|^2\}(G_1+iG_2),$$

in the sense of tight convergence[21] (G_1 and G_2 being two independent real standard Gaussian rv).

Proof (see Ref. 13 for a complete proof): Here (29) results directly from Lemma 2.18. For (30), the idea of the proof consists in the study of the sequence of rv $Z_p/p^{(n-2)/(2n)}$ [i.e., $\mathrm{Re}(Z_p)/p^{(n-2)/(2n)}$ and $\mathrm{Im}(Z_p)/p^{(n-2)/(2n)}$], and then, to conclude via the classical central limit theorem.

Corollary 2.20: More generally, let $(X_i)_{i\in\mathbb{N}^*}$ be a sequence of independent real positive rv with the same distribution, verifying

$$0<E\{X_1^n\}\quad\text{and}\quad E\{X_1^{n+1}\}<+\infty.$$

Then, if $(\epsilon_{[n]}^{(i)})_{i\in\mathbb{N}^*}$ denotes a sequence of independent Rademacher rv of order n and independent of the X_i, one has $T_p=p^{-1/n}(\Sigma_{i=1}^p\epsilon_{[n]}^{(i)}X_i)$, which converges weakly $(p\to+\infty)$, in the previous sense to T, with T verifying

$$T\overset{\text{Law}}{\sim}E^{1/n}\{X_1^n\}G_{[n]}+Y,$$

where $G_{[n]}$ is a standard n-Gaussian rv independent of Y, and Y is a $\mathbb{C}$-valued rv satisfying

$$(\forall\quad k\in\mathbb{N}^*),\quad E\{Y^k\}=E\{\overline{Y}^k\}=0,$$

$$E\{|Y|^2\}=+\infty,$$

$$Y\overset{\text{Law}}{\sim}\omega_q Y,\quad\text{for all } q=0,...,n-1.$$

Proof: That is an extension of the classical central limit theorem. In the case of signed or complex measures, the result has already been proved by Refs. 9, 22, and 2. Some expansions are viewed in Ref. 13 and, in particular, the study of the Bernoulli series of order n, i.e., of the form $S_p=\Sigma_{k=1}^p\alpha_k\epsilon_{[n]}^{(k)}$, where $(\alpha_k)_{k\in\mathbb{N}^*}$ is a sequence of $l_n(\mathbb{R}_+)$ and $(\epsilon_{[n]}^{(k)})_{k\in\mathbb{N}^*}$ is a sequence of independent Rademacher rv of order n.

Now, one shows an interesting asymptotic behavior for the $|G_{[n]}|$'s law, which is given by the following.

Lemma 2.21: ($\exists \quad B_n>0$), such that

$$(\forall \quad x\geqslant 0), \quad \mathbb{P}\{|G_{[n]}|>x\}\leqslant \exp\{-B_n x^{n'}\};$$

$$(n^{-1}+n'^{-1}=1).$$

Proof: Let $\gamma_n=n/[(n-1)!]^{1/(n-1)}$. One has $|G_{[n]}| \overset{\text{Law}}{\sim} \Pi_{j=0}^{n-2}\mu_{n,j}$ (see Ref. 13), where $(\mu_{n,j})_{0\leqslant j<n-2}$ is a family of independent real positive rv defined by

$$(\forall \quad x\geqslant 0),$$

$$\mathbb{P}\{\mu_{n,j}\leqslant x\}=\frac{n}{\gamma_n^{(j+1)/n}\Gamma((j+1)/n)}\int_0^x u^j \exp\left\{\frac{-u^n}{\gamma_n}\right\}du.$$

One verifies easily[13] that all the moments of $|G_{[n]}|$ and $\Pi_{j=0}^{n-2}\mu_{n,j}$ are identical, and also the characteristics functions and the laws.

Noting that $C_{n,j}=n/(\gamma_n^{(j+1)/n}\Gamma((j+1)/n))$, one deduces that

$$\mathbb{P}\{|G_{[n]}|>x\}=\prod_{j=0}^{n-2} C_{n,j}\int_{x_j}^{+\infty} u^j\exp\left\{\frac{-u^n}{\gamma_n}\right\}du$$

$$\leqslant (n-1)!\prod_{j=0}^{n-2}\int_{x_j}^{+\infty} u^{n-1} \times\exp\left\{\frac{-u^n}{\gamma_n}\right\}du,$$

with $\Pi_{j=0}^{n-2}x_j=x$, because the following functions F_j are positive on $\mathbb{R}_+$:

$$F_j(x)=\int_x^{+\infty}[C_{n,n-1}u^{n-1}-C_{n,j}u^j]\exp\{-u^n/\gamma_n\}du.$$

Indeed,

$$F_j(x)=\int_0^{+\infty}[C_{n,n-1}u^{n-1}-C_{n,j}u^j]\exp\{-u^n/\gamma_n\}du - \int_0^{x}[C_{n,n-1}u^{n-1}-C_{n,j}u^j]\exp\{-u^n/\gamma_n\}du.$$

From that, one deduces $F_j'(x)\geqslant 0$ for $0\leqslant x\leqslant (C_{n,n-1}/C_{n,j})^{1/(n-1-j)}$ and $F_j'(x)\leqslant 0$ for $x\geqslant (C_{n,n-1}/C_{n,j})^{1/(n-1-j)}$; one has, equally, $F_j(0)=0$ and $F_j(x)\geqslant 0$ for $x\geqslant 1$; thus $F_j(x)\geqslant 0$ for all $x\geqslant 0$.

Finally, one gets

$$\mathbb{P}\{|G_{[n]}|>x\}\leqslant (n-1)!\left\{\prod_{j=0}^{n-2}\frac{\gamma_n}{n}\exp\left(\frac{-x_j^n}{\gamma_n}\right)\right\}$$

$$\leqslant \exp\left\{\frac{-\Sigma_{j=0}^{n-2}x_j^n}{\gamma_n}\right\}.$$

With the help of the concavity of the Log function, one gets

$$(\forall \quad m\in\mathbb{N}^*) \quad (\forall \quad w_1,\ldots,w_n\in\mathbb{R}_+^*),$$

$$\text{Log}\{(w_1+\cdots+w_m)/m\}\geqslant\frac{1}{m}\{\text{Log } w_1+\cdots+\text{Log } w_m\}.$$

Hence one has $\mathbb{P}\{|G_{[n]}|>x\}\leqslant\exp\{[-(n-1)/\gamma_n]\times x^{n/(n-1)}\}$. Detailed results are given in Ref. 18 on densities similar to $p_n(x)$.

Remark 2.22: The studied sums of rv are not infinitely divisible in the Doob sense,[3] because $G_{[n]}$ is a $\mathbb{R}_+^{1/n}$-valued rv, whereas the sums of rv are $\mathbb{C}$ valued. Indeed, one has studied convolutions of measures defined on $\mathbb{R}_+^{1/n}$, which are constructed on $(\mathbb{R}_+^{1/n})^p$. Furthermore, all the given results are derived from the possible construction of the $n-\mathbb{R}_+$-isotropic part ϕ_0 of an analytic function in $\mathbb{C}$, ϕ, defined by

$$\phi_0(x)=\frac{1}{n}\left(\sum_{k=0}^{n-1}\phi(\omega_k x)\right), \quad (x\in\mathbb{R}).$$

Formally, we have used results on the lacunary series of order n, i.e., $S_p^{(n)}(z)=\Sigma_{k=0}^p a_{nk}z^k$ from the study of the entire series

$$S_p(\omega_r z)=\sum_{k=0}^p a_k(\omega_r z)^k, \quad \text{for } r=0,\ldots,n-1.$$

In Ref. 13, weak convergence is shown for S_p on the $\mathbb{C}\to\mathbb{C}$) continuous and bounded functions, by using an approximation on $l_n(\mathbb{R}^{1/n})$.

III. THE STANDARD *N*-BROWNIAN MOTION BASED ON $\mathbb{R}_+^{1/n}$

One gives a definition of this process allowing his construction, following Levy's ideas,[4,23] generalized at order n.

Definition 3.1: Given a complete space of probability $(\Omega,F,\mathbb{P})$, where F is the σ algebra $\otimes\ [B(\mathbb{R}_+^{1/n})]^{\mathbb{R}_+}$, with a filtration $(F_t,\ t\geqslant 0)$, one calls standard n-Brownian motion based on $\mathbb{R}_+^{1/n}$, F_t adapted, all the stochastic processes noted down $X_{[n]}=\{(X_{[n]}(t);t\geqslant 0):\mathbb{R}_+\to\mathbb{C}\}$ defined on $[\mathbb{C}U\{\infty\}]^{\mathbb{R}_+}$, by the following assertions:

$$X_{[n]}(0)=0, \quad \text{a.s} \tag{32a}$$

$(\forall \quad 0\leqslant s<t<\infty)$,

$$X_{[n]}(t)-X_{[n]}(s) \tag{32b}$$

is a $\mathbb{C}$-valued rv independent of F_s (in fact, $X_{[n]}$ has independent increments);

$(\forall \quad 0\leqslant s<t<+\infty)$,

$$X_{[n]}(t)-X_{[n]}(s)\overset{\text{law}}{\sim}(t-s)^{1/n}Z, \tag{32c}$$

$$\left(Z=w-\lim_{p\to+\infty} p^{-1/n}\left(\sum_{k=1}^{p} G_{[n]}^{(k)}\right) \quad \text{(see Sec. II)};\right.$$

$X_{[n]}$ has stationary and self-similar increments$\Big)$;

$$w-\lim_{t\to s^+}[(X_{[n]}(t)-X_{[n]}(s))/(t-s)^{1/n}]$$

exists as an n-Gaussian rv, (32d)

[the differential increments of $X_{[n]}$ are $\mathbb{R}_+^{1/n}$ valued and of order $(dt)^{1/n}$].

Remark 3.2: The no-respect of the condition (32d) would imply a no-isonomy of all the processes, verifying the other assertions (see the structure of the σ algebra). Furthermore, (32d) gives the fractal nature of this process that is totally different from the fractional Brownian motion of Lévy–Mandelbrot,[10] because his disjoint increments are independent and his variance is infinite. Now one gives a construction of $X_{[n]}$ by a linear cut approximation (time discretization).

Lemma 3.3: Let $(Z_q(t);\ t\in[0;T])_{q\in\mathbb{N}^*}$ $(0<T<+\infty)$ be the sequence of linear cut processes defined from the uniform partition of $[0;T]$ $\{kT/q\}$; $k=0,\dots,q-1$, and the family of independent standard n-Gaussian rv $\{G_{[n]}\times(r/q)\}_{r/q\in[0;1]\cap\mathbb{Q}}$, by the following assertions:

$$Z_q\left(\frac{kT}{q}\right)=\left(\frac{T}{q}\right)^{-1/n}\left\{\sum_{r=0}^{k-1} G_{[n]}\left(\frac{r}{q}\right)\right\},$$

$$\text{for all } k=0,\dots,q-1. \tag{33a}$$

If $t\in[kT/q;(k+1)T/q[$, then

$$Z_q(t)=Z_q\left(\frac{kT}{q}\right)+\frac{(t-kT/q)}{(T/q)}[Z_q((k+1)T/q)$$

$$-Z_q(kT/q)]. \tag{33b}$$

Then, $w-\lim_{q\to+\infty}Z_q(t)$ exists in the sense of Lemma 2.14, and has the same law as $t^{1/n}Z$ [see (32c) for the definition of Z].

Proof: From his construction $(\forall \quad 0\leqslant s<t<+\infty)$, $w-\lim_{q\to+\infty}[Z_q(t) - Z_q(s)]$ exists in the previous weak sense of Lemma 2.14 (see Ref. 13 for a detailed proof), and Z_q has independent separated increments. Next, one will show that the rv $Z_q(t)$ $(q\in\mathbb{N}^*)$ are tight for a fixed $t\in[0;T]$ (relative compacity in the sense of the Prohorov theorem[21]). First, one needs to know the behavior of $\mathbb{P}\{|Z_q(t)-Z_q(s)|>\epsilon\}$.

Lemma 3.4: $(\forall \quad q\in\mathbb{N}^*)$ $(\forall \quad \epsilon,\delta>0)$,

$$\mathbb{P}\{\sup_{|t-s|<\delta}|Z_q(t)-Z_q(s)|>\epsilon\}\leqslant\exp\{-B_n\delta^{-1/(n-1)}\epsilon^{n'}\}. \tag{34}$$

Proof:

$$\mathbb{P}\{|Z_q(t)-Z_q(s)|>\epsilon\}=\mathbb{P}\left\{\left|Z_q\left(\frac{k_1T}{q}\right)-Z_q\left(\frac{k_2T}{q}\right)+\left(t-\frac{k_1T}{q}\right)\left(\frac{T}{q}\right)^{1/n-1}G_{[n]}\left(\frac{k_1}{q}\right)-\left(s-\frac{k_2T}{q}\right)\left(\frac{T}{q}\right)^{1/n-1}\right.\right.$$

$$\left.\left.\times G_{[n]}\left(\frac{k_2}{q}\right)\right|>\epsilon\right\},$$

noting that $k_1=[qt/T]$ and $k_2=[qs/T]$. Then,

$$\mathbb{P}\{|Z_q(t)-Z_q(s)|>\epsilon\}=\mathbb{P}\left\{\left(\frac{T}{q}\right)^{1/n}\left|\left(\sum_{r=k_2}^{k_1-1} G_{[n]}\left(\frac{r}{q}\right)\right)+\left[\left(\frac{qt}{T}-k_1\right)G_{[n]}\left(\frac{k_1}{q}\right)-\left(\frac{qs}{T}-k_2\right)G_{[n]}\left(\frac{k_2}{q}\right)\right]\left(\frac{T}{q}\right)^{-1}\right|>\epsilon\right\}$$

$$\leqslant\mathbb{P}\left\{\left(\sum_{r=k_2}^{k_1-1}\left|G_{[n]}\left(\frac{r}{q}\right)\right|\right)+\left[\left(\frac{qt}{T}-k_1\right)\left|G_{[n]}\left(\frac{k_1}{q}\right)\right|+\left(\frac{qs}{T}-k_2\right)\left|G_{[n]}\left(\frac{k_2}{q}\right)\right|\right]\right.$$

$$\left.\times\left(\frac{T}{q}\right)^{-1}>\epsilon\left(\frac{T}{q}\right)^{-1/n}\right\}.$$

Thus,

$$\mathbb{P}\{|Z_q(t)-Z_q(s)|>\epsilon\}$$

$$\leqslant\left\{\prod_{r=k_2}^{k_1-1}\mathbb{P}\left[\left|G_{[n]}\left(\frac{r}{q}\right)\right|>\alpha_r\right]\right\}$$

$$\times\mathbb{P}\left\{\left(\frac{qt}{T}-k_1\right)\left(\frac{T}{q}\right)^{-1}\left|G_{[n]}\left(\frac{k_1}{q}\right)\right|>\beta_1\right\}$$

$$\times\mathbb{P}\left\{\left(\frac{qs}{T}-k_2\right)\left(\frac{T}{q}\right)^{-1}\left|G_{[n]}\left(\frac{k_2}{q}\right)\right|>\beta_2\right\},$$

with $\beta_1+\beta_2+\Sigma_{r=k_2}^{k_1-1}\alpha_r=\epsilon(T/q)^{-1/n}$.

By using the asymptotic estimation (31) on $\mathbb{P}\{|G_{[n]}|>x\}$, one gets

$$P\{|Z_q(t)-Z_q(s)|>\epsilon\}$$

$$\leqslant\left\{\prod_{j=k_2}^{k_1-1}\exp(-B_n\alpha_j^{n/(n-1)})\right\}$$

$$\times\exp\left\{-B_n\left(\frac{\beta_1(T/q)}{(qt/T-k_1)}\right)^{n/(n-1)}\right\}$$

$$\times\exp\left\{-B_n\left(\frac{\beta_2(T/q)}{(qs/T-k_2)}\right)^{n/(n-1)}\right\}.$$

That is a problem of variations with constraints, and one must minimize $\Sigma_{k=1}^{p}z_k^{n'}$ under the constraint $\Sigma_{k=1}^{p}\alpha_k z_k=x$ $(n'=n/(n-1))$. For that, one notes that

$$F(z_1,\ldots,z_p)=\sum_{k=2}^{p}z_k^{n'}+\alpha_1^{-n'}\left(x-\sum_{k=2}^{p}\alpha_k z_k\right)^{n'};$$

the minima of F are reached for $\partial F/\partial z_i=0$, $i=2,\ldots,p$; then the linear system in z_i,

$$\sum_{\substack{k=2\\k\neq 1}}^{p}\alpha_k z_k+\left(\alpha_i+\left(\frac{\alpha_1^{n'}}{\alpha_i}\right)^{1/(n'-1)}\right)z_i=x,\quad i=1,2,\ldots,p,$$

has a unique solution $z_j=x\alpha_j^{n-1}/(\Sigma_{k=1}^{p}\alpha_k^n)$, $j=1,\ldots,p$ (the determinant of the system is different from zero, and one gets the coordinates z_j of a certain gravity center): One deduces that

$$\sum_{k=1}^{p}z_k^{n'}\geqslant x^{n'}\frac{(\Sigma_{k=1}^{p}\alpha_k^{(n-1)n'})}{(\Sigma_{k=1}^{p}\alpha_k^n)}=x^{n'}\left(\sum_{k=1}^{p}\alpha_k^n\right)^{-1/(n-1)}$$

(this proof uses the same ideas as this one on Minkowski inequalities). Hence

$$\mathbb{P}\{|Z_q(t)-Z_q(s)|>\epsilon\}$$

$$\leqslant\exp\{-B_n(t-s)^{-1/(n-1)}\epsilon^{n'}\}.$$

Corollary 3.5: One has that $\lim_{\delta\to 0^+}\mathbb{P}\{\operatorname{Sup}_{|t-s|<\delta}|Z_q(t)-Z_q(s)|>\epsilon\}=0$, uniformly with q for all $\epsilon>0$, i.e., the rv $Z_q(t)$ are tight and converge weakly in the sense of $H_{n'}(\mathbb{C})$, to an n-Brownian standard motion based on $\mathbb{R}_+^{1/n}$, $X_{[n]}$.

Proof: From his construction, $X_{[n]}$ verifies (32a), because $Z_q(0)=0$, a.s for all $q\in\mathbb{N}^*$. Here

$$Z_q(t)-Z_q(s)=\left(\frac{qt}{T}-k_1\right)\left(\frac{T}{q}\right)^{1/n-1}G_{[n]}\left(\frac{k_1}{q}\right)$$

$$+\left(\frac{qs}{T}-k_2\right)\left(\frac{T}{q}\right)^{1/n-1}G_{[n]}\left(\frac{k_2}{q}\right)$$

$$+\left(\frac{T}{q}\right)^{1/n}\left[\sum_{r=k_2}^{k_1-1}G_{[n]}\left(\frac{r}{q}\right)\right].$$

This is a sum of independent standard n-Gaussian rv, and one gets

$$E\{(Z_q(t)-Z_q(s))^n\}=\left\{k_1-k_2+\left[\frac{(qt/T-k_1)}{(T/q)}\right]^n\right.$$

$$\left.+\left[\frac{(qs/T-k_2)}{(T/q)}\right]^n\right\}\left(\frac{T}{q}\right).$$

Since $qt/T-k_1<1$, $qs/T-k_2<1$, and $\lim_{q\to+\infty}(k_1-k_2)/(q/T)=t-s$, one deduces that

$$\lim_{q\to+\infty}E\{(Z_q(t)-Z_q(s))^n\}=t-s.$$

Via the results obtained in Sec. II, one deduces that

$$X_{[n]}(t)-X_{[n]}(s)\overset{\text{Law}}{\sim}(t-s)^{1/n}Z.$$

Since the family $\{G_{[n]}(r/q)\}_{0<r<q}$ with a fixed q is a family of independent standard n-Gaussian rv, and from the construction of Z_q, one deduces that for q great enough, $Z_q(t)-Z_q(s)$ is independent of $Z_q(t')-Z_q(s')$ for $0\leqslant s'<t'<s<t\leqslant T$; then, going to the limit $(q\to+\infty)$, one deduces that $X_{[n]}$ has separated independent increments. For (32d), one has an inversion limit problem, and one must study the ratio $\{[Z_q(t)-Z_q(s)]/(t-s)^{1/n}\}$; also, one comes down to the study of $E\{[Z_q(t)-Z_q(s)]^n/(t-s)\}$, because $E^{1/n}\{(\cdot)^n\}$ seems to be the adapted norm for the study of rv of type $S_p=\Sigma_{k=1}^{p}\alpha_k G_{[n]}^{(k)}$.

Corollary 3.6: Note that $X_{[n]}(t)=w-\lim_{q\to+\infty}Z_q(t)$ (in the sense defined previously); then w

$-\lim_{t\to s^+}[[X_{[n]}(t) - X_{[n]}(s)]/(t-s)^{1/n}]$ is a standard n-Gaussian rv, and also, $\mathbb{R}_+^{1/n}$ valued [in the sense of the convergence on $H_{n'}(\mathbb{C})$],

Proof:

$$E^{1/n}\left\{\left[\frac{X_{[n]}(t)-X_{[n]}(s)}{(t-s)^{1/n}}-G_{[n]}\left(\frac{k_2}{q}\right)\right]^n\right\}$$

$$\leqslant E^{1/n}\left\{\left[\frac{X_{[n]}(t)-X_{[n]}(s)-(Z_q(t)-Z_q(s))}{(t-s)^{1/n}}\right]^n\right\}$$

$$+E^{1/n}\left\{\left[\frac{Z_q(t)-Z_q(s)}{(t-s)^{1/n}}-G_{[n]}\left(\frac{k_2}{q}\right)\right]^n\right\}.$$

Also,

$$E^{1/n}\left\{\left[\frac{X_{[n]}(t)-X_{[n]}(s)}{(t-s)^{1/n}}-G_{[n]}\left(\frac{k_2}{q}\right)\right]^n\right\}$$

$$\leqslant\frac{(t-s)^{1/n}}{(t-s)^{1/n}}E^{1/n}\{Z^n\epsilon_q^n\}+E^{1/n}\left\{\left[\frac{(t-k_2T/q)}{(t-s)^{1/n}}\right.\right.$$

$$\left.\left.\times\left(\frac{T}{q}\right)^{1/n-1}G_{[n]}\left(\frac{k_2}{q}\right)-G_{[n]}\left(\frac{k_2}{q}\right)\right]^n\right\}.$$

Then, finally,

$$E^{1/n}\left\{\left[\frac{X_{[n]}(t)-X_{[n]}(s)}{(t-s)^{1/n}}-G_{[n]}\left(\frac{k_2}{q}\right)\right]^n\right\}$$

$$\leqslant\epsilon_q+\left|\frac{(t-k_2T/q)}{(t-s)^{1/n}}\left(\frac{T}{q}\right)^{1/n-1}-1\right|.$$

Because of the construction of $X_{[n]}$, $\lim_{q\to+\infty}\epsilon_q=0$ and for q large enough, one can choose $t-s$, verifying $t-s<T/q$ and $t-s\sim T/q\quad(q\to+\infty)$. Therefore

$$\lim_{q\to+\infty}\frac{(t-k_2T/q)}{(t-s)^{1/n}}\left(\frac{T}{q}\right)^{1/n-1}=1;$$

then, for $q=[T/(t-s)]$, one has

$$E^{1/n}\left\{\left[\frac{X_{[n]}(t)-X_{[n]}(s)}{(t-s)^{1/n}}-G_{[n]}\left(\frac{k_2}{q}\right)\right]^n\right\}\leqslant A(t-s)$$

[where A is a positive continuous function in 0, verifying $A(0)=0$].

Remark 3.7: To follow the Maruyama notation,[12,24,4] one writes, in a symbolic way, $X_{[n]}(t)=\int_0^t G_{[n]}(s)(ds)^{1/n}$ [where $\{G_{[n]}(s)\}_{(0\leqslant s\leqslant T)}$ is a family of independent standard n-Gaussian rv].

To obtain other results, one can consult Ref. 13. Next, one gives a construction of $X_{[n]}$ using the Ciesielski process,[23] and also, the Schauder functions.

Lemma 3.8: Let $\{G_{[n]}(k2^{-q});q\in\mathbb{N}^*;k\in\mathbb{N}^*;k\leqslant 2^q\}$ be a countable family of independent standard n-Gaussian rv, (independent only for different values of $k2^{-q}$). Then, one defines the following sequence of $\mathbb{C}$-valued linear cut processes $X_q=(X_q(t):0\leqslant t\leqslant 1)$ by (i) $X_1(t)=tG_{[n]}(1)$; (ii) X_q is linear in each interval $[(k-1)2^{-q};k2^{-q}]$ and continuous in t for all $\omega\in\Omega$; (iii) $X_{q+1}(2k2^{-(q+1)})=X_q(k2^{-q})$; and we see

$$\text{(iv)}\quad X_{q+1}((2k-1)2^{-(q+1)})$$

$$=X_q((2k-1)2^{-(q+1)})$$

$$+2^{-(1+q/n)}G_{[n]}((2k-1)2^{-(q+1)}).$$

We have the following results:

$$\mathbb{P}\{\lim_{q\to+\infty}X_q(t)\text{ exists for all }0\leqslant t\leqslant 1,$$

$$\text{uniformly in } t\}=1. \tag{35}$$

Proof:

$$\mathbb{P}\{\operatorname*{Max}_{t\in[0;1]}|X_{q+1}(t)-X_q(t)|\geqslant 2^{-q/(2n)}\}$$

$$=\mathbb{P}\{\operatorname*{Sup}_{1\leqslant k\leqslant 2^q}|G_{[n]}((2k-1)2^{-(q+1)})|2^{-(1+q/n)}$$

$$\geqslant 2^{-q/(2n)}\}.$$

Thanks to the asymptotic estimation on $\mathbb{P}\{|G_{[n]}|>x\}$ (31), one gets

$$\mathbb{P}\{\operatorname*{Sup}_{t\in[0;1]}|X_{q+1}(t)-X_q(t)|\geqslant 2^{-q/(2n)}\}$$

$$=\mathbb{P}\{\operatorname*{Sup}_{1\leqslant k\leqslant 2^q}|G_{[n]}((2k-1)2^{-(q+1)})|\geqslant 2^{(1+q/(2n))}\}$$

$$\leqslant 2^q\mathbb{P}\{|G_{[n]}^{(1)}|\geqslant 2^{(1+q/(2n))}\}$$

$$\leqslant 2^q\exp\{-(n-1)\gamma_n^{-1}2^{(1+q/(2n))n/(n-1)}\},$$

and *a fortiori*, $(\exists\quad q_0\in\mathbb{N}^*)$, such that $(\forall\quad q\geqslant q_0)$,

$$\mathbb{P}\{\operatorname*{Sup}_{t\in[0;1]}|X_{q+1}(t)-X_q(t)|\geqslant 2^{-q/(2n)}\}\leqslant 2^q2^{-2q}=2^{-q},$$

comparing the growths at the infinity between the functions x and $\exp(-x^\alpha)$ $(\alpha>0;\ x\to+\infty)$. Since, $\Sigma_{q\in\mathbb{N}}2^{-q}=2$ and $\Sigma_{q\in\mathbb{N}}2^{-q/(2n)}$ is convergent, also bounded by a certain $q_0'\in\mathbb{N}^*$, one deduces, via the Borel–Cantelli lemma, that for all $q>\operatorname{Sup}(q_0,q_0')$,

$$\mathbb{P}\{\lim_{k\to+\infty}(\operatorname*{Sup}_{t\in[0;1]}|X_k(t)-X_q(t)|)\geqslant 2^{-q/(2n)}q_0'\}\leqslant A_n2^{-q},$$

where A_n is a positive real constant depending only on n.

One remarks that the maxima of the processes X_q are growing with q and are attained at the top. In particular,

$$(\forall\epsilon>0),\ \lim_{q\to+\infty} \mathbb{P}\{\text{for } k>q \quad \text{and}$$

$$0\leqslant t\leqslant 1; |X_k(t)-X_q(t)|\geqslant\epsilon\}=0;$$

therefore the result and, *ipso facto*, one deduces the a.s continuity of the paths for the limit process $X_{[n]}$.

Corollary 3.9: The $X_{[n]}$'s paths are a.s continuous.

Lemma 3.10: Let $h^{[n]}_{k2^{-q}}(t)$ be the Schauder functions defined by

$$h^{[n]}_{k2^{-q}}(t)=0, \quad \text{for} \quad |t-k2^{-q}|\geqslant 2^{-q}, \tag{36a}$$

$$h^{[n]}_{k2^{-q}}(k2^{-q})=2^{-(q+1)/n}, \tag{36b}$$

$$h^{[n]}_{k2^{-q}} \text{ is linear in two adjacent intervals.} \tag{36c}$$

Then, the following series is convergent to an n-Brownian motion $X_{[n]}$ in the sense defined previously, and one can write $(\forall\ t\in[0;1])$,

$$X_{[n]}(t)=\sum_{q=0}^{+\infty}\ \sum_{\substack{k=1\\ k\text{ odd}}}^{2^q} h^{[n]}_{k2^{-q}}(t)G_{[n]}(k2^{-q}). \tag{37}$$

$$=\lim_{q\to+\infty} X_q(t) \quad \text{a.s.}$$

Proof: The independence of the separated increments is obtained by remarking that

$$X_q(t)-X_q(s)$$

$$=X_q(t)-X_q([2^qt]2^{-q})$$

$$+X_q(([2^qs]+1)2^{-q})-X_q(s)$$

$$+\sum_{m=[2^qs]+2}^{[2^qt]}[X_q(m2^{-q})-X_q((m-1)2^{-q})],$$

for $0\leqslant s<t\leqslant 1$.

For the construction of $X_q(t)-X_q(s)$, one uses only distinct $G_{[n]}(k2^{-q})$ of these used for the construction of $X_q(t')-X_q(s')$ $(0\leqslant s'<t'<s<t\leqslant 1)$; also, via the independence of the $G_{[n]}(k2^{-q})$, one gets the independence for separated increments, and the result is valid, going to the limit $(q\to+\infty)$. One has

$$X_q(t)-X_q(s)=2^{-q/n}\left\{\sum_{r[2^qs]+2}^{[2^qt]} G_{[n]}(r2^{-q})\right.$$

$$+(t-[2^qt]2^{-q})G_{[n]}(([2^qt]+1)2^{-q})$$

$$+(([2^qs]+1)2^q-s)G_{[n]}$$

$$\left.\times(([2^qs]+1)2^{-q})\right\}.$$

Also, $X_q(t)-X_q(s)$ is of type $\Sigma_k\alpha_kG^{(k)}_{[n]}$ $(\alpha_k\geqslant 0)$. Via an extension of the isometry with the cone $l_n(\mathbb{R}_+)$, one can find the limit law of $X_q(t)-X_q(s)$ $(q\to+\infty)$ in the weak sense, previously defined in Sec. II. One has

$$E\{[X_q(t)-X_q(s)]^n\}=2^{-q}\left\{\left(\sum_{r=[2^qs]+2}^{[2^qt]} 1\right)\right.$$

$$+(t-[2^qt]2_n^{-q})$$

$$\left.+(([2^qs]+1)2^{-q}-s)n\right\}$$

$$\to t-s, \quad (q\to+\infty)$$

and therefore

$$X_{[n]}(t)-X_{[n]}(s)\overset{\text{Law}}{\sim}(t-s)^{1/n}Z$$

(Z defined in Sec. II as the limit rv obtained in the central limit theorem at order n). But, one has the following "strange" properties.

Lemma 3.11:

$$\lim_{q\to+\infty} E\{|X_q(t)-X_q(s)|^2\}=+\infty \quad (0\leqslant s<t<+\infty), \tag{38a}$$

$(\forall\lambda\in\mathbb{C})$,

$$E\{\exp(\lambda[X_{[n]}(t)-X_{[n]}(s)])\}=\exp\{\lambda^n(t-s)/n!\}. \tag{38b}$$

$$E\{|X_{[n]}(t)-X_{[n]}(s)|^2\}=+\infty, \tag{38c}$$

$(\forall\quad \phi\in H_{n'}(\mathbb{C}))$,

$$E\{\phi(X_{[n]}(t)-X_{[n]}(s))\}$$

$$=\sum_{k=0}^{+\infty}\frac{\phi^{(nk)}}{k!}(0)(t-s)^k$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(u)\exp\left\{\frac{(-iu)^n(t-s)}{n!}\right\}du \tag{38d}$$

[where $\hat{\phi}(u)=\int_{\mathbb{R}}\exp(iux)\phi(x)dx$].

Proof:

$$E\{|X_q(t)-X_q(s)|^2\}\sim\frac{2^{q(1-2/n)}n\Gamma(2)}{(n!)^{2/n}\Gamma(2/n)}(t-s)\to+\infty,$$

$$(q\to+\infty).$$

The other results are directly validated through the previous constructions of $X_{[n]}$ and the properties of sums of n-Gaussian rv studied in Sec. II (i.e., the central limit theorem at order n).

Remark 3.12: Equation (38d) joins together $X_{[n]}$ and the Hochberg's process $X_{[2p]}^{(\mathrm{H})}$ [2] (that is defined only for $n=2p$ for convergence reasons), and that verifies the following.

For all functions ϕ_{H} ($\mathbb{R}\to\mathbb{R}$) in the Schwartz class,

$$E\{\phi_{\mathrm{H}}(X_{[2p]}^{(\mathrm{H})}(t)-X_{[2p]}^{(H)}(s))\}$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{\phi}(u)\exp\left\{-\frac{u^{2p}}{(2p)!}(t-s)\right\}du. \tag{39}$$

Lemma 3.13: Each of the following processes is a standard n-Brownian motion based on $\mathbb{R}_+^{1/n}$:

$$X_{[n]}(t)-X_{[n]}(s),\quad t,s\geqslant 0\quad \text{(Additivity property)}, \tag{40a}$$

$$\omega_k X_{[n]}(t),\quad t\geqslant 0,\quad k\in\mathbb{N}\quad (n\text{-}\mathbb{R}_+\text{ isotropy}), \tag{40b}$$

$$cX_{[n]}(t/c^n),\quad t\geqslant 0,\quad c>0\quad (n\text{-Brownian scaling}), \tag{40c}$$

$$t^{2/n}X_{[n]}(1/t),\quad t>0\quad (n\text{ inversion}), \tag{40d}$$

$$\omega_k h^{-1/n}[X_{[n]}(s+ht)-X_{[n]}(s)],$$

$$k\in\mathbb{N},\ h,t,s>0\ \text{(self-similarity)}. \tag{40e}$$

Proof: Equation (40a) is evident; Eq. (40b) results directly from the property established on Z in Sec. II. For Eqs. (40c) and (40d), an easy proof is given in Ref. 13; Eq. (40e) is the result of (40a), (40b), and (40c).

Remark 3.14: In Ref. 13, one shows the strong Markov property and the no-differentiability of the sample continuous paths of $X_{[n]}$, with the help of an evident generalization of the case $n=2$.[25] Next, one studies the variations of order n of the a.s continuous sample paths of $X_{[n]}$, which is the starting point for the definition of a stochastic integral and an Itô-Taylor lemma of order n, using an extension of the isometry on the cone $l_n(\mathbb{R}_+)$.

Lemma 3.15: Let $T>0$ and $\{t_{q,k}\}$ be a growing partition of $[0;T]$, more and more thin (i.e., $0=t_{q,0}<\cdots<t_{q,k}<\cdots<t_{q,q}=T$; $\lim_{q\to+\infty}\operatorname{Sup}_{0<k<q}|t_{q,k+1}-t_{q,k}|=0$); then, one has

$$\lim_{q\to+\infty}\sum_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^n \tag{41}$$

exists and is $\mathbb{R}_+$ valued [in the sense of quadratic mean and probability convergence; $\Delta_{q,k}X_{[n]}=X_{[n]}(t_{q,k+1})-X_{[n]}(t_{q,k})$].

Proof: Indeed, one has chosen $X_{[n]}$ among the processes, verifying $w-\lim_{t\to s^+}[[X_{[n]}(t)-X_{[n]}(s)]/(t-s)^{1/n}]$ is $\mathbb{R}_+^{1/n}$ valued; then, if the limit of $\Sigma_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^n$ exists, it is $\mathbb{R}_+$ valued, and to prove the quadratic mean convergence, it is sufficient to calculate $E\{((\Sigma_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^n)-T)^2\}$ in place of $E\{|(\Sigma_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^n)-T|^2\}$; therefore

$$E\left\{\left(\left(\sum_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^n\right)-T\right)^2\right\}$$

$$=E\left\{\left(\sum_{k=0}^{q-1}([\Delta_{q,k}X_{[n]}]^n-\Delta_{q,k}t)\right)^2\right\}$$

$$=\sum_{k=0}^{q-1}E\{([\Delta_{q,k}X_{[n]}]^n-\Delta_{q,k}t)^2\},$$

because $T=\Sigma_{k=0}^{q-1}\Delta_{q,k}t$ $[\Delta_{q,k}X_{[n]}]^n-\Delta_{q,k}t$ is independent of $[\Delta_{q,j}X_{[n]}]^n-\Delta_{q,j}t$ for $k\neq j$ and $E\{[\Delta_{q,k}X_{[n]}]^n-\Delta_{q,k}t\}=0$; furthermore,

$$E\{([\Delta_{q,k}X_{[n]}]^n-\Delta_{q,k}t)^2\}=E\{[\Delta_{q,k}X_{[n]}]^{2n}\}-E\{[\Delta_{q,k}t]^2\};$$

then, one gets

$$E\left\{\left[\left(\sum_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^n\right)-T\right]^2\right\}$$

$$=\sum_{k=0}^{q-1}(E\{[\Delta_{q,k}X_{[n]}]^{2n}\}-E\{[\Delta_{q,k}t]^2\})$$

$$=\left[\frac{(2n)!}{2!(n!)^2}-1\right]\left(\sum_{k=0}^{q-1}[\Delta_{q,k}t]^2\right)$$

$$<\left[\frac{(2n)!}{2!(n!)^2}-1\right] T(\sup_{0<k<q}[\Delta_{q,k}t])\to 0$$

$$(q\to+\infty).$$

Corollary 3.16: $(\forall \quad q\in\mathbf{N}^*)$,

$$E\left\{\left(\left(\sum_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^n\right)-T\right)^2\right\}$$

$$=\left[\frac{(2n)!}{2!(n!)^2}-1\right]\left(\sum_{k=0}^{q-1}[\Delta_{q,k}t]^2\right)>0. \qquad (42)$$

Proof: $[(2n-1)(2n-2)\cdots(n+1)]/[(n-1)(n-2)\cdots 2\times 1]>1$ for $n>2$; then the result is validated.

Via the Markov–Tchebychev inequality, one deduces the convergence in probability for $\Sigma_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^n$ $(q\to+\infty)$, and for a dyadic partition, one has the a.s convergence to T (in fact, that is the law of great numbers).

Corollary 3.17: More generally, for a partition $\{t_{q,k}\}_{0<k<q}$ of $[s;t]$ as the previous one defined in Lemma 3.15, one has the following energy identity at order n, i.e.,

$$E\{[X_{[n]}(t)-X_{[n]}(s)]^n\}=\sum_{k=0}^{q-1}E\{[\Delta_{q,k}X_{[n]}]^n\}. \qquad (43)$$

Proof: The proof uses the independence of the separated $X_{[n]}$'s increments and the values of the different moments of $\Delta_{q,k}X_{[n]}$ up to the order n, i.e.,

$$E\{[\Delta_{q,k}X_{[n]}]^j\}=\delta_{j,n}[\Delta_{q,k}t], \quad \text{for} \quad j=1,2,...,n.$$

Notation 3.18: Formally, one notes that

$$\int_0^t [dX_{[n]}(s)]^j = w-\lim_{q\to+\infty}\sum_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^j \qquad (44)$$

(in the sense of the strictly definite positive functional $E\{(\cdot)^l\}$, with $l=\inf_{p\in\mathbf{N}^*}\{np/j\in\mathbf{N}^*\}$.

One then deduces, with the help of the stochastic integral defined independently in Sec. IV, the following recurrence relations.

Lemma 3.19: One has

$$\int_0^t [dX_{[n]}(s)]^j = X_{[n]}^j(t)-\sum_{r=1}^{j-1}C_j^r\int_0^t X_{[n]}^{j-r}(s)$$

$$\times[dX_{[n]}(s)]^r. \qquad (45)$$

Proof: This identity is true, even going to the limit $(q\to+\infty)$ on the partition $\{t_{q,k}\}_{0<k<q}$, and it uses only the binomial identity. Furthermore, $\int_0^t[dX_{[n]}(s)]^j$ defines a complex-valued stochastic process with differential increments as $\mathbb{R}_+^{j/n}$ valued $(\mathbb{R}_+^{j/n}=\{z\in\mathbb{C};z=\rho\omega_{jk};\ \rho\in\mathbb{R}_+;\ 0\leqslant k\leqslant n-1\})$.

Corollary 3.20: Noting that $p_0=rj/n$ and $r=\inf_{p\in\mathbf{N}^*}\{np/j\in\mathbf{N}^*\}$, one gets the following.

$$\text{If } p_0\neq 1,\ \int_0^t [dX_{[n]}(s)]^j=0 \text{ in the sense of } E\{(\cdot)^r\}. \qquad (46)$$

Proof: Indeed, $[\Delta_{q,k}X_{[n]}]^j\sim[\Delta_{q,k}t]^{j/n}G_{[n]}^j(t_{q,k})$ $(q\to+\infty)$, and therefore the differential increments are $\mathbb{R}_+^{j/n}$ valued. Then, if $p_0\neq 1$, one has

$$E\left\{\left(\sum_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^j\right)^r\right\}$$

$$=\sum_{k_1,...,k_r=0}^{q-1}E\{[\Delta_{q,k_1}X_{[n]}]^j\cdots[\Delta_{q,k_r}X_{[n]}]^j\}.$$

Therefore

$$E\left\{\left(\sum_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^j\right)^r\right\}=\sum_{k=0}^{q-1}E\{[\Delta_{q,k}X_{[n]}]^{np_0}\}$$

$$=\frac{(np_0)!}{p_0!(n!)^{p_0}}\left(\sum_{k=0}^{q-1}[\Delta_{q,k}t]^{p_0}\right)>0,$$

and going to the limit $(q\to+\infty)$, one has

$$E\left\{\left(\sum_{k=0}^{q-1}[\Delta_{q,k}X_{[n]}]^j\right)^r\right\}$$

$$<\frac{(np_0)!}{p_0!(n!)^{p_0}}t(\sup_{0<k<q}[\Delta_{q,k}t]^{p_0-1})\to 0 \quad (q\to+\infty).$$

Corollary 3.21: In the case $p_0=1$, i.e., $n/j\in\mathbf{N}^*$,

$$w-\lim_{q\to+\infty}\sum_{k=0}^{N(t,q)-1}[\Delta_{q,k}X_{[n]}]^j=\int_0^t[dX_{[n]}(s)]^j \qquad (47)$$

is a standard n/j-Brownian motion, $[N(t,q)$ is defined by $t_{q,N(t,q)}\leqslant t<t_{q,N(t,q)+1}]$.

Proof: One studies the sums of rv of type $Z_r^{(q)}(t)=\Sigma_{k=0}^{N(t,q)-1}[\Delta_{q,k}X_{[n]}]^j$, where $\{t_{q,k}\}_{0<k<q}$ is a growing partition of $[0;T]$ more and more thin.

Because of the independence of the $X_{[n]}$'s increments, one gets easily

$$(\forall \quad 0\leqslant s<t\leqslant T) \quad (\forall \quad m=1,...,r-1)$$

$$E\{[Z_r^{(q)}(t)-Z_r^{(q)}(s)]^m\}=0,$$

and by construction, $Z_r^{(q)}$ has independent increments, and therefore

$$E\{[Z_r^{(q)}(t)-Z_r^{(q)}(s)]^r\}=t_{q,N(t,q)}-t_{q,N(q,s)}\to t-s \quad (q\to+\infty).$$

Furthermore,

$$\Delta_{q,k}Z_r^{(q)}=[\Delta_{q,k}X_{[n]}]^j\sim[\Delta_{q,k}t]^{1/r}G^{\,j}_{[n]}(t_{q,k}) \quad (q\to+\infty).$$

Using the previous results on the central limit theorem at order r given in Sec. II and the construction of the standard r-Brownian motion based on $\mathbb{R}_+^{1/r}$, one deduces that $Z_r^{(q)}$ converges weakly to a standard r-Brownian motion. One can see Ref. 13 for the details of the proof and the characteristic properties of $X_{[n]}$.

Remark 3.22: One remarks that the process,

$$Z(t)=X_1(X_0^+(t))+iX_2(X_0^-(t)), \quad (t\geqslant 0)$$

verifies

$$(\forall \quad 0<s<t<+\infty) \quad (\forall \quad \lambda\in\mathbb{C}),$$

$$E\{\exp(\lambda[Z(t)-Z(s)])\}$$
$$=E_{X_0}\{E_{X_1,X_2}\{\exp(\lambda[X_1(X_0^+(t))-X_1(X_0^+(s))+i(X_2(X_0^-(t))-X_2(X_0^-(s)))])\}\}$$
$$=E_{X_0}\left\{\exp\left(\frac{\lambda^2}{2}[X_0^+(t)-X_0^+(s)]\right)\times\exp\left(-\frac{\lambda^2}{2}[X_0^-(t)-X_0^-(s)]\right)\right\}$$
$$=E_{X_0}\left\{\exp\left(\frac{\lambda^2}{2}[X_0(t)-X_0(s)]\right)\right\}=\exp\left\{\frac{\lambda^4}{8}(t-s)\right\} \tag{48}$$

[where X_0, X_1, and X_2 are classical standard and independent real Brownian motions, $X_0^+(t)=\mathrm{Sup}(X_0(t),0)$, and $X_0^-(t)=-\mathrm{Inf}(X_0(t),0)$, E_X being the average over X].

But, the increment $Z(t)-Z(s)$ is dependent of $\{Z(u);\ 0\leqslant u\leqslant s\}$. To a multiplicative constant less, the increments $Z(t)-Z(s)$ have the same extension of Fourier transform as the increments of $X_{[4]}(t)-X_{[4]}(s)$, i.e., $E\{\exp(\lambda[Z(t)-Z(s)])\}=\exp\{(\lambda^4/8)(t-s)\}$ and $E\{\exp(\lambda[X_{[4]}(t)-X_{[4]}(s)])\}=\exp\{(\lambda^4/24)(t-s)\}$.

The E_{X_1,X_2} partial moments are as unbounded, considering $X_0^+(t)$ and $X_0^-(t)$ as random times related, respectively, to X_1 and X_2; Z is a fractional Brownian motion of order $1/n$,[10] and one can say that it is constructed from the standard real Brownian motion by a stochastic iteration.

Conclusion 3.23: Finally, one remarks that $X_{[n]}$ has been constructed from the definition of a discrete n martingale,[13] i.e., from the following process.

Definition 3.24: Let $(\Omega,F,\mathbb{P})$ be a probability space with a filtration $(F_k)_{k\in\mathbb{N}}$; then, the $\mathbb{C}$-valued discrete stochastic process, $M=(M_k)_{k\in\mathbb{N}}$, F_k adapted, is a discrete n martingale, if and only if

$$M(0)=0, \quad \text{a.s}, \tag{49a}$$

$$(\forall \quad k\in\mathbb{N}), \quad |E\{M_k^n\}|<+\infty, \tag{49b}$$

$$(\forall \quad k\in\mathbb{N}) \quad [M_{k+1}-M_k]^n \text{ is a real positive rv}, \tag{49c}$$

$$(\forall \quad 0\leqslant j\leqslant k) \quad (\forall \quad p=1,\dots,n-1),$$

$$E\{(M_k^p-M_j^p)/F_j\}=0. \tag{49d}$$

IV. A STOCHASTIC INTEGRAL AND AN ITÔ–TAYLOR LEMMA OF ORDER *N*

Preliminary 4.1: One gives an elementary construction of the stochastic integral of order n in relation to $X_{[n]}$, by using an extension of the isometry between the cone of rv (it is a discrete n martingale), of type $\Sigma_k\alpha_k G_{[n]}^{(k)}$ [where $(\alpha_k)_{k\in\mathbb{N}^*}\in l_n(\mathbb{R}_+)$ and $\{G_{[n]}^{(k)}\}_{k\in\mathbb{N}^*}$ is a family of independent standard n-Gaussian rv], and the cone $l_n(\mathbb{R}_+)$ via the functional $E^{1/n}\{(\cdot)^n\}$ definite strictly positive on this type of sums of rv; then, one deduces an Itô–Taylor lemma of order n.

Definition 4.2: Let $E_{[n]}^+([0;T])$ $(0<T<+\infty)$, denote the set of step functions $([0;T]\times\Omega\to\mathbb{R}_+)$, F_t adapted and a.s bounded; then, one defines for all functions $f^+(u,\omega)\in E_{[n]}^+([0;T])$, the following stochastic integral:

$$\int_0^t f^+(u,\omega)dX_{[n]}(u) = \sum_{k=0}^{p-1} f^+(t_k,\omega)[X_{[n]}(t_{k+1})-X_{[n]}(t_k)], \tag{50}$$

[where $\{t_k\}$ is a partition of $[0;T]$, attached to f^+ by: $(\forall \quad u\in[t_k;t_{k+1}[)$, $f^+(u,\omega)=f^+(t_k,\omega)$].

Lemma 4.3: For all $j=1,\dots,n-1$ $(\forall \quad 0\leqslant s\leqslant t\leqslant T)$, one has

$$E\left\{\left(\int_s^t f^+(u,\omega)dX_{[n]}(u)\right)^j\Big/F_s\right\}=0, \tag{51a}$$

$$E\left\{\left(\int_s^t f^+(u,\omega)dX_{[n]}(u)\right)^n \Big/ F_s\right\}$$

$$=E\left\{\int_s^t (f^+(u,\omega))^n\,du \Big/ F_s\right\}. \tag{51b}$$

Proof: ($\forall \quad j=1,...,n-1$),

$$E\left\{\left(\int_s^t f^+(u,\omega)dX_{[n]}(u)\right)^j \Big/ F_s\right\}$$

$$=E\left\{\left(\sum_{k=0}^{p-1} f^+(t_k,\omega)\left[X_{[n]}(t_{k+1})-X_{[n]}(t_k)\right]\right)^j \Big/ F_s\right\}$$

$$=\sum_{k=0}^{p-1} E\{[f^+(t_k,\omega)]^j/F_{t_k}\}\,E\{[X_{[n]}(t_{k+1})$$

$$-X_{[n]}(t_k)]^j/F_{t_k}\}=0,$$

because for $j=1,...,n-1$, $E\{[X_{[n]}(t_{k+1})-X_{[n]}(t_k)]^j/F_{t_k}\}=0$, $f^+(t_k,\omega)$ is F_{t_k} adapted and

$$E\{[X_{[n]}(t_{k_1+1})-X_{[n]}(t_{k_1})]^{m_1}\cdots[X_{[n]}(t_{k_r+1})$$

$$-X_{[n]}(t_{k_r})]^{m_r}/F_s\}=0,$$

for all $r=2,...,j$; $m_1+\cdots+m_r=j$ and $k_1<k_2<\cdots<k_r$.

When $j=n$, it remains only that

$$\sum_{k=0}^{p-1} E\{[f^+(t_k,\omega)]^n/F_{t_k}\}\,E\{[X_{[n]}(t_{k+1})$$

$$-X_{[n]}(t_k)]^n/F_{t_k}\}$$

$$=\sum_{k=0}^{p-1} E\{(f^+(t_k,\omega))^n/F_s\}(t_{k+1}-t_k)$$

$$=E\left\{\int_s^t (f^+(u,\omega))^n\,du/F_s\right\}.$$

Lemma 4.4: Noting that $M^+_{[n]}([0;T])$ the set of random functions $f^+(t,\omega)$ $([0;T]\times\Omega\to\mathbb{R}_+)$, verifying $E\{\int_0^T(f^+(u,\omega))^n\,du\}<+\infty$, for all $f^+(t,\omega)\in M^+_{[n]}([0;T])$, one defines $\int_0^t f^+(u,\omega)dX_{[n]}(u)$, $t\in[0;T]$ [approximating f^+ by an a.s growing sequence of functions $g_p^+\in E^+_{[n]}([0;T])$], F_t adapted in the sense of the following convergence:

$$\lim_{p\to+\infty} E\left\{\int_0^T [f^+(t,\omega)-g_p^+(t,\omega)]^n\,dt\right\}=0.$$

Proof: Using the arguments known and developed at order 2,[3,26] one deduces the existence of the stochastic integral of order n by the extension of the isometry on $L_n(d\mathbb{P}\times dt)$, which is given by the strictly definite positive functional $E\{(\cdot)^n\}$.

Lemma 4.5: One defines $\int_0^t X^k_{[n]}(s)dX_{[n]}(s)$ for all $t\in[0;T]$ and $k\in\mathbb{N}$ by a time discretization. Then, one goes to the limit on the time step in the sense of the functional $E\{(\cdot)^n\}$ (for the convergence).

Proof: $\int_0^t X^k_{[n]}(s)dX_{[n]}(s)$ is also defined by the extension of an isometry first, considering the sums of rv of type $\Sigma_{j=0}^{q-1}X^k_{[n]}(t_{q,j})[\Delta_{q,j}X_{[n]}]$, where $\{t_{q,j}\}$ is a growing partition of $[0;t]$, more and more thin, and remarking that

$$E\left\{\left(\sum_{j=0}^{q-1} X^k_{[n]}(t_{q,j})[\Delta_{q,j}X_{[n]}]\right)^n\right\}$$

$$=\sum_{j=0}^{q-1} E\{X^{kn}_{[n]}(t_{q,j})\}E\{[\Delta_{q,j}X_{[n]}]^n/F_{t_{q,j}}\}$$

$$=\frac{(nk)!}{k!(n!)^k}\left(\sum_{j=0}^{q-1} t^k_{q,j}[\Delta_{q,j}t]\right)$$

$$\to\frac{(nk)!}{k!(n!)^k}\left(\int_0^t u^k\,du\right)\quad(q\to+\infty).$$

Corollary 4.6: One has

$$E\left\{\left(\int_0^t X^k_{[n]}(s)dX_{[n]}(s)\right)^n\right\}=\frac{(nk)!t^{k+1}}{(k+1)!(n!)^k}. \tag{52}$$

Corollary 4.7: Noting that $S_q=\Sigma_{j=0}^{q-1}X^k_{[n]}(t_{q,j})[\Delta_{q,j}X_{[n]}]$, one obtains

$$(\forall \quad q\in\mathbb{N}),\quad E\{(S_{q+1}-S_q)^n\}>0, \tag{53a}$$

$$(\forall \quad v\in\mathbb{N}),\quad \lim_{q\to+\infty} E\{(S_{q+v}-S_q)^n\}=0;$$

$$\operatorname*{Sup}_{q\in\mathbb{N}} E\{S_q^n\}<+\infty, \tag{53b}$$

S_q is convergent in the sense of $E\{(\cdot)^n\}$. (53c)

Proof: In order to construct $\{t_{q+1,j}\}$ from $\{t_{q,j}\}$, we put on a real t_{q+1,j_0+1} verifying

$$t_{q,j_0}=t_{q+1,j_0}<t_{q+1,j_0+1}<t_{q,j_0+1}=t_{q+1,j_0+2}.$$

Note that $t_1=t_{q,j_0}$, $t_2=t_{q+1,j_0+1}$, $t_3=t_{q,j_0+1}$. Then, one has

$$S_{q+1}-S_q=-X^k_{[n]}(t_1)[X_{[n]}(t_3)-X_{[n]}(t_1)]+X^k_{[n]}(t_1)\times[X_{[n]}(t_2)-X_{[n]}(t_1)]+X^k_{[n]}(t_2)[X_{[n]}(t_3)-X_{[n]}(t_2)]=[X_{[n]}(t_3)-X_{[n]}(t_2)][X^k_{[n]}(t_2)-X^k_{[n]}(t_1)].$$

Therefore, by using the binomial formula written under the form

$$a^k-b^k=(b+a-b)^k-b^k=\sum_{r=1}^{k} C^r_k b^{k-r}(a-b)^r \quad (a,b\in\mathbb{C}),$$

one obtains

$$E\{(S_{q+1}-S_q)^n\}=E\{[X_{[n]}(t_3)-X_{[n]}(t_2)]^n[X^k_{[n]}(t_2)-X^k_{[n]}(t_1)]^n\}=E\left\{\left(\sum_{r=1}^{k} C^r_k X^{k-r}_{[n]}(t_1)[\Delta_{q+1,j_0}X_{[n]}]^r\right)^n[\Delta_{q+1,j_0+1}X_{[n]}]^n\right\}=\left(\sum_{r_1,\dots,r_n=1}^{k} C^{r_1}_k\cdots C^{r_n}_k E\{X^{nk-(r_1+\cdots+r_n)}_{[n]}(t_1)\}E\{[\Delta_{q+1,j_0}X_{[n]}]^{r_1+\cdots+r_n}\}\right)E\{[\Delta_{q+1,j_0+1}X_{[n]}]^n\}>0.$$

Then, using the $X_{[n]}$'s and $\Delta X_{[n]}$'s moment values, one obtains

$$E\{(S_{q+1}-S_q)^n\}=[\Delta_{q+1,j_0+1}t]\times\left(\sum_{\substack{r_1,\dots,r_n=1\\ r_1+\cdots+r_n=nm}}^{k} C^{r_1}_k\cdots C^{r_n}_k\times\frac{[n(k-m)]!}{(k-m)!(n!)^{k-m}}t_1^{k-m}\frac{(nm)!}{m!(n!)^m}[\Delta_{q+1,j_0}t]^m\right).$$

Finally, one carries out step by step, adding an intermediary point at each iteration, and one gets

$$E\{(S_{q+1}-S_q)^n\}=E\left\{\left(\sum_{w=1}^{v}\left(\sum_{r=1}^{k} C^r_k X^{k-r}_{[n]}(t_{q+w,j_0})\times[\Delta_{q+w,j_0}X_{[n]}]^r\right)\times(\Delta_{q+w,j_0+1}X_{[n]})\right)^n\right\}.$$

To develop this expression, one remarks that one obtains products depending of the parameter $R\leqslant n$ of the following form:

$$E\left\{\prod_{\substack{i=1\\ \left(\substack{z_1+\cdots+z_R=n\\ 1\leqslant w_1<\cdots<w_R\leqslant v}\right)}}^{R}\left(\sum_{r=1}^{k} C^r_k X^{k-r}_{[n]}(t_{q+w_i,j_0})\times[\Delta_{q+w_i,j_0}X_{[n]}]^r\right)^{z_i}[\Delta_{q+w_i,j_0+1}X_{[n]}]^{z_i}\right\}.$$

In such a product denoted by P_R, $[\Delta_{q+w_R,j_0+1}X_{[n]}]^{z_R}$ is independent of the rest of P_R, for $1\leqslant z_R\leqslant n$; then $P_R=0$ if $z_R\neq n$, because

$$E\{[\Delta_{q+w_R,j_0+1}X_{[n]}]^k\}=\delta_{k,n}[\Delta_{q+w_R,j_0+1}t],\quad \text{for } k=1,\dots,n.$$

Therefore $E\{(S_{q+v}-S_q)^n\}$ has the following simplified expression:

$$E\{(S_{q+v}-S_q)^n\}=\sum_{w=1}^{v} E\left\{\left(\sum_{r=1}^{k} C^r_k X^{k-r}_{[n]}(t_{q+w,j_0})[\Delta_{q+w,j_0}X_{[n]}]^r\right)^n\times[\Delta_{q+w,j_0+1}X_{[n]}]^n\right\}=\sum_{w=1}^{v}[\Delta_{q+w,j_0+1}t]$$

$$\times\left(\sum_{\substack{r_1,\ldots,r_n=1\\(r_1+\cdots+r_n=nm)}}^{k} C_k^{r_1}\cdots C_k^{r_n}\frac{[n(k-m)]!(nm)!}{(k-m)!m!(n!)^k}\times t_{q+w,j_0}^{k-m}[\Delta_{q+w,j_0}t]^m\right)>0,$$

for all $q,v\in\mathbf{N}^*$; furthermore, one gets

$$E\{(S_{q+v}-S_q)^n\}\leqslant K(T)\left(\sum_{w=1}^{v}[\Delta_{q+w,j_0}t]^2\right)\leqslant K'(T)(\sup_{0\leqslant j<q}[\Delta_{q,j}t])\to 0\quad(q\to+\infty).$$

Also, one has an order on the S_q in the sense of the functional $E\{(\cdot)^n\}$, and since S_q is bounded in the sense of $E\{(\cdot)^n\}$, one has a kind of dominated convergence theorem and S_q converges in the sense of $E\{(\cdot)^n\}$ [S_q is a Cauchy sequence in the sense of $E\{(\cdot)^n\}$].

Corollary 4.7: From the existence of stochastic integrals of type $\int_0^t X_{[n]}^k(s)dX_{[n]}(s)$, one deduces the existence of stochastic integrals of the following type: $\int_0^t f(X_{[n]}(s))dX_{[n]}(s)$, where f verifies f and f^n are analytic in $\mathbb{C}$

$$f^n(z)=\sum_{k=0}^{+\infty}a_k z^k\quad(a_k,z\in\mathbb{C}),\tag{54a}$$

$$\lim_{k\to+\infty}\frac{k^{n-2}a_{n(k+1)}}{a_{nk}}=0.\tag{54b}$$

Proof:

$$E\left\{\left[\int_0^t f(X_{[n]}(s))dX_{[n]}(s)\right]^n\right\}=t\left[\sum_{k=0}^{+\infty}\frac{(nk)!a_{nk}}{(k+1)!}\left(\frac{t}{n!}\right)^k\right],$$

and the condition given in the corollary is sufficient to have the convergence of this series. The details of the calculus are given in Ref. 13.

Remark 4.8: One can define a "local" stochastic integral, i.e., $\int_0^{t\wedge T_a}f(X_{[n]}(s))dX_{[n]}(s)$ with the help of the stopping time $T_a=\inf\{T>0;|X_{[n]}(T)|\geqslant a\}$ $(a>0)$, for an analytic function f in $B(0,a)=\{z\in\mathbb{C};\ |z|<a\}$ (where $f^n(z)=\sum_{k=0}^{+\infty}a_kz^k$ is verified: the entire series $S(z)=\sum_{k=0}^{+\infty}[(nk)!a_{nk}/(k+1)!](z/n!)^k$ has a convergence radius $R>T$ $(T<+\infty)$).

Remark 4.9: Other properties of the stochastic integral of order n are given in Ref. 13. Next, one examines the construction of the Itô–Taylor lemma of order n, which will be written under a symbolic form (semigroup and random drift) with the following formula:

$$d[f(X_{[n]}(t))]=\left\{\exp\left(dX_{[n]}(t)\frac{d.}{dz}\right)-Id\right\}f(X_{[n]}(t))=\sum_{k=1}^{n}\frac{1}{k!}f^{(k)}(X_{[n]}(t))[dX_{[n]}(t)]^k\tag{55}$$

(where f is in the previous class of analytic functions in $\mathbb{C}$ in a first approach).

Lemma 4.10: $w-\lim_{q\to+\infty}\sum_{r=0}^{q-1}[\Delta_{q,r}X_{[n]}]^k$ is denoted by $\int_0^t[dX_{[n]}(s)]^k$ (with the previous conventions and notations, the convergence being defined in the sense of the functional $E\{(\cdot)^{np_0/k}\}$, $p_0=\inf_{p\in\mathbf{N}^*}\{np/k\in\mathbf{N}^*\}$).

If $p_0=1$, i.e., $n/k\in\mathbf{N}^*$, then

$$\int_0^t[dX_{[n]}(s)]^k=X_{[n/k]}(t)\tag{56a}$$

is an n/k-Brownian motion.

If $p_0>1$, then

$$\int_0^t[dX_{[n]}(s)]^k=0.\tag{56b}$$

Proof: It is sufficient to show the result just in the case $k>n$. One also has

$$\left|\sum_{r=0}^{q-1}[\Delta_{q,r}X_{[n]}]^k\right|\leqslant t(\sup_{0\leqslant r<q}|\Delta_{q,r}X_{[n]}|^{k-n})\to 0\quad(q\to+\infty),$$

since the sample paths of $X_{[n]}$ are a.s continuous and $\{t_{q,r}\}$ is a growing partition of $[0;t]$ more and more thin.

Lemma 4.11: For all $m\in\mathbf{N}^*$, one has

$$X_{[n]}^m(t)=\sum_{k=1}^{m}\frac{1}{k!}\int_0^t\left\{\frac{d^k}{dx^k}[X_{[n]}^{m-k}(s)]\right\}[dX_{[n]}(s)]^k.\tag{57}$$

Proof: Using the binomial formula and the discretization in time of the stochastic integral, one gets

$$X_{[n]}^m(t_{q,r+1})=[X_{[n]}(t_{q,r})+\Delta_{q,r}X_{[n]}]^m=\sum_{k=0}^{m}C_m^kX_{[n]}^{m-k}(t_{q,r})[\Delta_{q,r}X_{[n]}]^k;$$

therefore

$$X^m_{[n]}(t_{q,r+1}) - X^m_{[n]}(t_{q,r}) = \sum_{k=1}^{m} C^k_m X^{m-k}_{[n]}(t_{q,r})[\Delta_{q,r} X_{[n]}]^k,$$

and summing on r, it comes to

$$X^m_{[n]}(t) = \sum_{k=1}^{m} C^k_m \left\{ \sum_{r=0}^{q-1} X^{m-k}_{[n]}(t_{q,r})[\Delta_{q,r} X_{[n]}]^k \right\};$$

then going to the limit $(q \to +\infty)$, one obtains with the previous notations

$$X^m_{[n]}(t) = \sum_{k=1}^{m} C^k_m \int_0^t X^{m-k}_{[n]}(s)[dX_{[n]}(s)]^k$$
$$= \sum_{k=1}^{m} \frac{1}{k!} \int_0^t \left\{ \frac{d^k}{dx^k}[X^{m-k}_{[n]}(s)] \right\} [dX_{[n]}(s)]^k.$$

Lemma 4.12: For all functions f analytic in $\mathbb{C}$ verifying the previous conditions, one has

$$f(X_{[n]}(t)) - f(0) = \sum_{k=1}^{n} \frac{1}{k!} \int_0^t f^{(k)}(X_{[n]}(s)) \times [dX_{[n]}(s)]^k, \tag{58a}$$

i.e., with the differential notation

$$d[f(X_{[n]}(t))] = \left\{ \exp\left(dX_{[n]}(t) \frac{d.}{dz} \right) - Id \right\} (f(X_{[n]}(t)))$$
$$= \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(X_{[n]}(t))[dX_{[n]}(s)]^k. \tag{58b}$$

Proof: The Itô–Taylor lemma has been proved for the polynomial functions of type $P(x) = x^m$ $(m \in \mathbb{N})$ in Lemma 4.11; therefore it is proved for the polynomial functions with complex coefficients. To deduce this lemma on some analytic functions, one uses the Stone–Weierstrass theorem of approximation of an holomorphic function in a compact set of $\mathbb{C}$ by a sequence of polynomial functions in the sense of the uniform convergence. First, let f be analytic on $\mathbb{C}$ and $\{P_{m,a}\}_{m\in\mathbb{N}}$ $(a>0)$, a sequence of polynomial functions approximating f on the compact set $B(0,a)$: One has $(\forall \quad z \in B(0,a))$,

$$P_{m,a}(z) = f(0) + zf'(0) + \cdots + \frac{z^{n-1}}{(n-1)!} f^{(n-1)}(0)$$
$$+ \int_0^z \frac{(z-z')^{n-1}}{(n-1)!} P^{(n)}_{m,a}(z') dz'.$$

Via the Itô–Taylor lemma of order n on $P_{m,a}$, one obtains

$$P_{m,a}(X_{[n]}(t)) - P_{m,a}(0)$$
$$= \sum_{k=1}^{n} \frac{1}{k!} \int_0^t P^{(k)}_{m,a}(X_{[n]}(s)) dX_{[n/k]}(s).$$

Then, going to the limit $(m \to +\infty)$, even if one might stop $X_{[n]}$ on $B(0,a)$ with the help of the stopping time T_a, one gets

$$f(X_{[n]}(t \wedge T_a)) - f(0)$$
$$= \sum_{k=1}^{n} \frac{1}{k!} \left\{ \lim_{m \to +\infty} \int_0^{t \wedge T_a} P^{(k)}_{m,a}(X_{[n]}(s)) dX^{(s)}_{[n/k]} \right\}.$$

Since

$$\lim_{m \to +\infty} E\left\{ \int_0^{T \wedge T_a} \left| P^{(k)}_{m,a}(X_{[n]}(s)) - f^{(k)}(X_{[n]}(s)) \right|^{n/k} ds \right\}$$
$$= 0,$$

one deduces the following result on the stochastic integrals:

$$\lim_{m \to +\infty} \int_0^{t \wedge T_a} P^{(k)}_m(X_{[n]}(s))[dX_{[n]}(s)]^k$$
$$= \int_0^{t \wedge T_a} f^{(k)}(X_{[n]}(s)) dX_{[n/k]}(s),$$

in the weak sense given previously, defining the stochastic integral for all $a < +\infty$. Since $\lim_{a \to +\infty} T_a = +\infty$, one then goes to the limit $(a \to +\infty)$, on each side of the identity. Also, one obtains the Itô–Taylor lemma of order n for the analytic functions in $\mathbb{C}$, verifying

$$\left| E\left\{ \int_0^T [f^{(n)}(X_{[n]}(s))]^n ds \right\} \right| < +\infty,$$

which gives the same type of existence condition as that obtained for the construction of the stochastic integral of order n of $f^{(n)}$.

Remark 4.13: In Ref. 13, one goes further, studying the diffusions of order n of type

$$d\xi(t) = \sum_{i=1}^{n} a_i(t,\omega) dX^{(i)}_{[i]}(t),$$

where $\{X^{(i)}_{[i]}\}_{1<i<n}$ is a family of independent i-Brownian motions. One therefore obtains an Itô–Taylor lemma on these diffusions.

V. RESOLUTION OF HEAT PROBLEMS OF ORDER N; MEASURES CONSTRUCTED FROM $\mathbb{R}_+^{1/n}$

Definition 5.1: The heat polynomials of order n,[5,6,27,28] $W_{n,p}(t,x)$ are defined by the generating function $\exp(xz+tz^n)$, i.e.,

$(\forall \quad x,z,t\in\mathbb{C})$,

$$\exp(xz+tz^n)=\sum_{p=0}^{+\infty} W_{n,p}(t,x)\frac{z^p}{p!}, \tag{59}$$

and by an identification on the formal series, one obtains

$$\begin{aligned} W_{n,p}(t,x)&=p!\left\{\sum_{k+nr=p}\frac{x^k}{k!}\frac{t^r}{r!}\right\}\\ &=E\{(x+(n!t)^{1/n}G_{[n]})^p\}\\ &=E\{(x+(n!)^{1/n}X_{[n]}(t))^p\} \end{aligned} \tag{60}$$

(where $G_{[n]}$ is a standard n-Gaussian rv and $X_{[n]}$ is a standard n-Brownian motion).

Corollary 5.2: $W_{n,p}(t,x)$ satisfies the following heat problem of order n:

$$\frac{\partial u}{\partial t}(t,x)=\frac{\partial^n u}{\partial x^n}(t,x), \quad (t,x)\in[0;T]\times\mathbb{C},$$

$$u(0,x)=x^p \quad (x\in\mathbb{R}). \tag{61}$$

Proof:

$$\begin{aligned} \frac{\partial W_{n,p}}{\partial t}(t,x)&=p!\left(\sum_{k+nr=p}\frac{x^k}{k!}r\frac{t^{r-1}}{r!}\right)\\ &=p!\left(\sum_{k+nr=p}\frac{x^k}{k!}\frac{t^{r-1}}{(r-1)!}\right), \end{aligned}$$

$$\begin{aligned} \frac{\partial^n W_{n,p}}{\partial x^n}(t,x)&=p!\left(\sum_{q+nr=p}\frac{x^q}{q!}\frac{t^r}{r!}\right)\\ &=p!\left(\sum_{q+nm=p}\frac{x^q}{q!}\frac{t^{m-1}}{(m-1)!}\right), \end{aligned}$$

with $m=r+1$ and $q=k-n$. Thus the result is proved.

Corollary 5.3: More generally, let

$$g_n(\lambda)=\exp\left\{\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\lambda^k x_k\right\} \quad (\lambda,x_k\in\mathbb{C})$$

and $\{H_p(x_1,...,x_n)\}_{p\in\mathbb{N}}$ denote the sequence of polynomials with n variables defined by

$$g_n(\lambda)=\sum_{p=0}^{+\infty} H_p(x_1,...,x_n)\lambda^p,$$

i.e.,

$$H_p=\frac{1}{p!}\left\{\frac{\partial^p g_n}{\partial\lambda^p}(\lambda)\right\}_{\lambda=0}.$$

Then $(\forall \quad p\in\mathbb{N})$,

$$\begin{aligned} &H_p\left(\int_0^t dX_{[n]}(s),\int_0^t [dX_{[n]}(s)]^2,...,\int_0^t [dX_{[n]}(s)]^n\right)\\ &\quad=\int_0^t dX_{[n]}(t_1)\int_0^{t_1} dX_{[n]}(t_2)\cdots\int_0^{t_{p-1}} dX_{[n]}(t_p) \end{aligned} \tag{62}$$

(in the sense of the stochastic integral of order n iterated at order p).

Proof: It is sufficient to verify that

$$W_\lambda(t)=\exp\left\{\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\int_0^t \lambda^k[dX_{[n]}(s)]^k\right\}$$

is the unique solution of the stochastic differential equation of order n:

$$dM_\lambda(t)=M_\lambda(t)dX_{[n]}(t),$$

$$M_\lambda(0)=1.$$

Thus, writing the present formal series, one obtains the following recurrence:

$$Z_p(t)=\int_0^t Z_{p-1}(s)dX_{[n]}(s),$$

where

$$Z_p(t)=H_p\left(\int_0^t dX_{[n]}(s),...,\int_0^t [dX_{[n]}(s)]^n\right).$$

For the details and an extension of this result, one can see Ref. 13.

Remark 5.4: Hochberg[2] has already given the beginnings of this result, and via the Itô–Taylor lemma, the $H_p(x_1,...,x_n)$ are explicitly defined.

One can equally notice that Jumarie[29] already linked $g_n(\lambda)$ to the following evolution PDE of order n:

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x)=\sum_{k=1}^{n}\frac{a_k^k}{k!}\frac{\partial^k u}{\partial x^k}(t,x) \quad (t,x)\in[0;T]\times\mathbb{C}\\ &u(0,x)=f(x) \quad (x\in\mathbb{C}) \end{aligned} \tag{63}$$

(where f is a given function with "good" analytic properties).

He uses an entropy notion. To solve this problem, one constructs stochastic processes called semi-martingales of order n[13] in place of $X_{[n]}$, defined by

$$dX(t)=\sum_{k=1}^{n} a_k\, dX_{[k]}^{(k)}(t),$$

with $X(0)=0$ a.s, $a_k \in \mathbb{C}$ and $\{X_{[k]}^{(k)}\}_{1<k<n}$ a family of independent k-Brownian motions.

Lemma 5.5: Let f be a function verifying the conditions of the Itô–Taylor lemma of order n (Lemma 4.12). Then, one gets the following Dynkin's formula: of order n, i.e., $(\forall \quad \alpha_n \in \mathbb{C})$, $(\forall \quad \Delta t \in \mathbf{R}^*)$,

$$E\{f(x+\alpha_n X_{[n]}(t+\Delta t))\}-E\{f(x+\alpha_n X_{[n]}(t))\}$$
$$=\frac{1}{n!}\int_t^{t+\Delta t} \alpha_n^n E\{f^{(n)}(x+\alpha_n X_{[n]}(s))\} ds. \qquad (64)$$

Proof: The proof given in Ref. 13 uses the Itô–Taylor lemma of order n (Lemma 4.12), i.e.,

$$f(x+\alpha_n X_{[n]}(t+\Delta t))-f(x+\alpha_n X_{[n]}(t))$$
$$=\sum_{k=1}^{n}\frac{1}{k!}\int_t^{t+\Delta t} \alpha_n^k f^{(k)}(x+\alpha_n X_{[n]}(s))[dX_{[n]}(s)]^k.$$

Then, going to the average on each side of the identity, it comes out that

$$E\{f(x+\alpha_n X_{[n]}(t+\Delta t))-f(x+\alpha_n X_{[n]}(t))\}$$
$$=\frac{1}{n!}\int_t^{t+\Delta t} \alpha_n^n E\{f^{(n)}(x+\alpha_n X_{[n]}(s))\} ds,$$

since all the other averages are null for all $x\in\mathbb{C}$, $t\geqslant 0$, $\Delta t\in\mathbf{R}^*$.

Lemma 5.6: The heat equation of order n,

$$\frac{\partial u}{\partial t}(t,x)=\frac{\alpha_n^n}{n!}\frac{\partial^n u}{\partial x^n}(t,x), \quad (t,x)\in[0;T]\times\mathbb{C}$$
$$u(0,x)=f(x) \quad (x\in\mathbb{C};\alpha_n\in\mathbb{C}^*) \qquad (65)$$

(where f is a $(\mathbf{R}\to\mathbf{R})$ function, continuous in $L_1(\mathbb{R},dx)$ with a holomorphic extension to $\mathbb{C}$ defined by $\tilde{f}(z)=1/2\pi\int_{\mathbb{R}}\hat{f}(\lambda)\exp(-i\lambda z)d\lambda$, with

$$\hat{f}(\lambda)=\int_{\mathbb{R}}\exp(i\lambda x)f(x)dx \quad (\lambda\in\mathbb{R})),$$

has a solution given by

$$u(t,x)=E\{\tilde{f}(x+\alpha_n X_{[n]}(t))\}$$
$$=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\lambda)\exp\left\{-i\lambda x+(-i\alpha_n\lambda)^n\frac{t}{n!}\right\}d\lambda. \qquad (66)$$

Proof: Going to the limit $(\Delta t\to 0)$ in Lemma 5.5, one deduces immediately the result.

Remark 5.7: There is a relation between the class of admissible functions, α_n and n. For n even, one recognizes the Hochberg's result,[2] and one can choose $\alpha_n=\exp\{i\pi(1/n-\frac{1}{2})\}$, i.e., $\alpha_n^n=-(-i)^n$ to obtain the biggest class of admissible functions. If one chooses $\alpha_n=\exp\{(i\pi/2)(1/n-1)\}$, i.e., $\alpha_n^n=i^{n-1}$, always for n even, it is sufficient to have $\hat{f}(x)\in L_1(\mathbb{R},dx)$; also one can take f in $L_2\cap L_1$, by using the Plancherel identity. For n odd and $\alpha_n=1$, one can take equally $f\in L_2\cap L_1$. One can equally remark that the formula (66) gives the solution with the help of a pseudodifferential operator (Parseval identity, I Ref. 13).

Extension 5.8: Using the same method as in Lemma 5.5 with a stopping time, one obtains a Dynkin formula of order n with a stopping time and in Lemma 5.6 one can extend the result to holomorphic functions f in B_a $(a>0)$ defined by $B_a=\{z\in\mathbb{C};|\operatorname{Im} z|<a\}$. By a more general way, one approaches f on a compact set of $\mathbb{R}$ by a sequence of polynomials $\{P_q\}_{q\in\mathbf{N}}$ in the sense of the uniform convergence, and the formal solution is given by

$$u(t,x)=\lim_{q\to+\infty} E\{P_q(x+\alpha_n X_{[n]}(t))\}. \qquad (67)$$

This solution is admissible for n even, $\alpha_n=\exp\{i\pi(1/n-\frac{1}{2})\}$ and $f\in L_2\cap L_1$ and continuous in $\mathbb{R}$.

Appendix 5.9: In Ref. 13, one studies more generally the PDE of order n via the notion of stochastic differential equations of order n, in the scalar and finite-dimensional case, which will be the subject of a next paper. Moreover, one will develop the notion of discrete and continuous Martingales of order n. Next, one gives some informations on basic measures implicitly used in this paper to define the standard n-Brownian motion. We will content ourselves to describe here the case $n=3$.

Definition 5.10: Let ν denote a probability measure on $(\mathbb{R},B(\mathbb{R}))$, with a density $\nu'(x)$ in relation with the Lebesgue measure defined on $(\mathbb{R},B(\mathbb{R}))$. From ν, one defines the measure μ on $(\mathbb{C},B(\mathbb{R}_+^{1/3}))$ by the following assertions:

$B(\mathbb{R}_+^{1/3})$ is the σ algebra generated by sets of the following type:

$$i_{1/3}(x,h)=\{z\in\mathbb{C};(z-x)^3\in[0;h^3[\}, \text{ where } (x,h)\in\mathbb{R}\times\mathbb{R}_+^*. \qquad (68a)$$

Let $I_3=\{i_{1/3}(x,h);(x,h)\in\mathbb{R}\times\mathbb{R}_+^*\}$,

and denote by $[x;x+h^3[=(i_{1/3}(x,h))^3$. (68b)

Then, μ verifies $(\forall\ \ A,B\in I_3)$,

$$\mu(A)=\nu(A^3), \tag{69a}$$

$$\mu(A\cap B)=\nu(A^3\cap B^3). \tag{69b}$$

Lemma 5.11: Immediately, one deduces that

$$(\forall\ \ A,B\in I_3),\quad \mu(A\cup B)=\nu(A^3\cup B^3). \tag{70}$$

Proof:

$$\mu(A\cup B)=\mu(A)+\mu(B)-\mu(A\cap B)$$

$$=\nu(A^3)+\nu(B^3)-\nu(A^3\cap B^3)$$

$$=\nu(A^3\cup B^3).$$

Remark 5.12: First,

$$\mathbb{C}=\bigcup_{n\in\mathbb{N}^*}\ \bigcup_{x_i\in\mathbb{Q}} i_{1/3}(x_i,n^{1/3}),$$

and then, by using (70), one has

$$\mu(\mathbb{C})=\nu\Big(\bigcup_{n\in\mathbb{N}^*}\ \bigcup_{x_i\in\mathbb{Q}}[x_i;x_i+n[\Big)=\nu(\mathbb{R})=1. \tag{71}$$

Equally, one gets the following.

Lemma 5.13:

$$\mu(\mathbb{R})=1\quad\text{and}\quad\mu(\mathbb{C}/\mathbb{R})=0. \tag{72}$$

Proof:

$$\mathbb{R}=\bigcap_{q\in\mathbb{N}^*}\ \bigcup_{x_i\in\mathbb{Q}} i_{1/3}(x_i,q^{-1/3})$$

because $\lim_{q\to+\infty} q^{-1/3}=0$, and then the branches $[x_i;x_i+\omega_k q^{-1/3}[$ $[(k=1,2)$ have a length tending to 0 $(q\to+\infty)$, while $\mathbb{R}\subset\bigcup_{x_i\in\mathbb{Q}} i_{1/3}(x_i,q^{-1/3})$ for all $q\in\mathbb{N}^*]$. Then

$$\mu(\mathbb{R})=\mu\Big(\bigcap_{q\in\mathbb{N}^*}\ \bigcup_{x_i\in\mathbb{Q}} i_{1/3}(x_i,q^{-1/3})\Big)$$

$$=\lim_{q\to+\infty}\ \mu\Big(\bigcup_{x_i\in\mathbb{Q}} i_{1/3}(x_i,q^{-1/3})\Big)$$

$$=\lim_{q\to+\infty}\ \nu\Big(\bigcup_{x_i\in\mathbb{Q}}[x_i;x_i+q^{-1}]\Big)$$

$$=\lim_{q\to+\infty}\ \nu(\mathbb{R})=1.$$

Then one obtains $\mu(\mathbb{C}/\mathbb{R})=0$.

Lemma: 5.14: More generally, let $a,b\in\mathbb{R}$ $(a<b)$; then

$$\mu([a;b[)=\nu([a;b[). \tag{73}$$

Proof: One uses the previous method with

$$[a;b[=\bigcap_{q\in\mathbb{N}^*}\ \bigcup_{x_i\in\mathbb{Q}\cap[a;b[} i_{1/3}(x_i,q^{-1/3}).$$

Lemm a 5.15: $(\forall\ \ F\in B(\mathbb{R}_+^{1/3}))$,

$$\mu(F)=0\Rightarrow\nu(F\cap\mathbb{R})=0. \tag{74}$$

Then, the parts of $B(\mathbb{R}_+^{1/3})$ that do not contain a no empty interval of $\mathbb{R}$ are negligible in the Lebesgue sense.

Lemma 5.16: Let

$$J_q=\bigcup_{k=0}^{q-1} i_{1/3}(a+(k/q)(b-a),\ [(b-a)/q]^{1/3}),$$

with $[(b-a)/q]<1$ and $q\in\mathbb{N}^*$. Then

$$\mu(J_q)=\nu([a;b[), \tag{75}$$

and going to the limit $(q\to+\infty)$, one will write formally

$$\frac{d\mu}{(d\nu)^{1/3}}=1 \tag{76}$$

[in the sense of the Radon–Nikodym measure derivative on $B(\mathbb{R}_+^{1/3})$ for μ and on $B(\mathbb{R})$ for ν].

Remark: 5.17: Via nonstandard analysis, all the results developed here can be rewritten following the ideas of Keisler[11] (he has constructed a nonstandard Itô calculus of order 2 with the help of an infinitesimal step time discretization).

VI. CONCLUSION

We have given some indications for the development of a stochastic analysis of order n $(n>2)$ and their applications to the study of evolution PDE of order n. The singular properties of the sets $A(0,h)$ generating $B(\mathbb{R}_+^{1/n})$ conduct to a "strange" behavior, in comparison with the standard real Brownian motion of Levy. In a future paper, we will develop the notions taken up here in the more general context of martingales and diffusions of order n. Applications will be given to solve the master equation and others.[24,30,31] Finally, one remarks that Zachary[32] has implicitly used measures based on $\mathbb{R}_+^{1/n}$ to solve the Gelfand–Levitan problem of order n.

[1] Block, Ark. Math., Astron. Fys. **7**, 13 (1912); **2**, 7 (1912); **3**, 1 (1912); **4**, 8 (1912).

[2] Hochberg, Ann. Probab. **6**, 433 (1978); Proc. Am. Math. Soc. **79**, 298 (1980).

[3] Doob, *Stochastic Processes* (Wiley, New York, 1953).

[4] Lévy, *Processus Stochastiques et mouvement Brownien* (Gauthier-Villars, Paris, 1965).
[5] Kemnitz, SIAM J. Math. Anal. **13**, 640 (1982); **16**, 379 (1985).
[6] Watzlawek, Appl. Anal. **18**, 283 (1984); M.h. Math. **100**, 67 (1985).
[7] Motoo, *Stochastic Analysis and Applications*, Advances in Probability, edited by Pinsky (Dekker, New York, 1984), Vol. 7.
[8] Nishioka, Jpn. J. Math. **11**, 59 (1985).
[9] Gaveau and Sainty, Lett. Math. Physics **15**, 345 (1988).
[10] Mandelbrot and Ness, SIAM Rev. **10**, 422 (1968).
[11] Keisler, Memoirs of the AMS, No. 297, 1984, Vol. 48.
[12] Berger and Sloan, *LNM 932* (Springer-Verlag, 1982), Vol. 1; Memoirs of the AMS, No. 266, 1982; J. Funct. Anal. (1983); J. Diff. Equations **57**, 15 (1985); **62**, 252 (1986).
[13] Sainty, Ph.D. thesis, University of Paris, 1987.
[14] Ungar, Am. Math. Month. **89**, 688 (1982).
[15] Van Hamme, Am. Math. Month. **1**, 70 (1984).
[16] Mittag-Leffler, Acta. Math. **29**, 101 (1904).
[17] Montel, Leçons sur les Fonctions Univalentes (Gauthier-Villars, Paris, 1933).
[18] Zolotarev, *Transactions of Mathematical Monographs* (American Mathematical Society, Providence, 1986), Vol. 65.
[19] Pollard, Bulletin of the American Math. Society 1948, p. 1115.
[20] Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1966), 2nd ed., Vols. 1 and 2.
[21] Billingsley, *Probability and Measure* (Wiley, New York 1979).
[22] Krylov, Sov. Math. Dokl. **1**, 760 (1960).
[23] Knight, Mathematical Surveys of the American Mechanical Society, No. 18, 1981.
[24] Chaturvedi and Gardiner, J. Stat. Phys. **17** 429 (1977); **18**, 501 (1978).
[25] Itô and McKean, *Diffusion Processes and their Sample Paths* (Springer-Verlag, New York, 1965), Band 125.
[26] Gihman and Skorohod, *Stochastic Differential Equations* (Springer-Verlag, New York 1972), Band 72.
[27] Feinsilver, *LNM 696* (Springer-Verlag, New York, 1978).
[28] Nevai, *2nd Conference on Approximation Theory* (American Mathematical Society, Providence, 1980), Vol. 3.
[29] Jumarie, J. Math. Phys. **26**, 1173 (1985).
[30] Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
[31] Dubinskii, Russ. Math. Surveys **37**, 5 (1982).
[32] Zachary, J. Math. Anal. Appl. **117**, 449 (1986).
[33] Diaz and Means, Ann. Mat. **97**, 113 (1973).
[34] Roetman, J. Diff. Equations **9**, 335 (1971).